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THE EQUIVALENCE OF DEFINITIONS OF A MATRIC FUNCTION

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1. Introduction. The general field of matric analysis, which is finding ever-increasing applicability in systems of linear differential equations [3], [13] and in quantum mechanics, rests upon the extension of the function concept to matrices. Two different basic approaches are available for such an extension. One may consider matrices whose elements are functions of a complex parameter z. From this standpoint one has a mapping of a set of complex numbers into the set of all matrices of a given order. On the other hand one may consider mappings of a set of matrices into a set of matrices. This second concept is the one with which this paper will be concerned.

We shall consider square matrices with elements in the complex field and define a general matric function to be a correspondence f which relates to each admissible matrix Z of order n a matrix W, denoted by $f(Z)$, of order n with elements in the complex field. What constitutes an admissible matrix Z depends of course upon the nature of the correspondence f. If $f(A)$ is uniquely determined by A, then f is said to be *single-valued* at A; otherwise it is multiple-valued at A.

Matric polynomials with scalar coefficients provide familiar examples of such matric functions. Any scalar polynomial $p(x)=\sum_{i=0}^{k} c_i x^i$ with complex coefficients gives rise to a matric polynomial $p(Z)$ by "substitution" of the matrix Z for the indeterminate x. By "substitution" we mean that Z^i is determined by matric multiplication, x^0 is replaced by the identity matrix of order n, $c_i Z^i$ is interpreted as scalar multiplication of Z^i by c_i and the addition as matric addition. Such matric polynomials are matric functions in the sense defined above. They further obey the same rational integral laws of combination as their scalar brothers.

It is natural to inquire whether non-polynomial functions (or more generally non-algebraic functions) of a complex variable similarly admit a matric counterpart or analogue.* Whether or not such a counterpart or extension of scalar functions is useful and interesting depends on what combinatorial properties of scalar functions are preserved in the matric function. Fantappiè [6] was the first writer to state explicitly a desirable set of combinatorial requirements to be fulfilled by a definition of such a matric function. These are the following, where $f(A)$ denotes the matric function arising from the scalar function $f(z)$:

I.	*If* $f(z)=k$,	*then* $f(A)=k\,I$,
II.	*If* $f(z)=z$,	*then* $f(A)=A$,
III.	*If* $f(z)=g(z)+h(z)$,	*then* $f(A)=g(A)+h(A)$,
IV.	*If* $f(z)=g(z)\cdot h(z)$,	*then* $f(A)=g(A)\,h(A)$,

* From now on the term "matric function" will be confined to mean a function arising from a scalar function of a complex variable.

where A is an admissible matrix for each of the functions, and where in III *and* IV *matric addition and multiplication are implied in $f(A)+g(A)$ and $f(A)\ h(A)$.**

These requirements will insure (1) that the definition, when applied to a polynomial $p(z)$, will yield the usual matric polynomial $p(A)$, and (2) that any rational identity in scalar functions of a complex variable will be fulfilled by the corresponding matric functions; *e.g.*, if $\sin A$ and $\cos A$ can be defined in accordance with the above, then $(\sin A)^2+(\cos A)^2=I$.

A fifth requirement that would be highly desirable would be

V. If $f(z)=h(g(z))$, then $f(A)=h(g(A))$,

for all admissible A. This requirement, if attainable, would insure that each *functional* identity among scalar functions would be preserved in their matric function counterparts; *e.g.*, from $e^{\log z}=z$, would follow $e^{\log A}=A$.

The extension of the concept of a function of a complex variable to matric functions has occupied the attention of a number of mathematicians since 1883. The history of the field is unusual in that many of the writers seem to have been unaware of what had been done by their predecessors. It seems that almost every mathematician who became intrigued by the idea proceeded to frame his own definition of a matric function, with little or no attention to connections with earlier definitions.

As a result there have been proposed in the literature since 1880 eight distinct definitions of a matric function, by Weyr, Sylvester and Buchheim, Giorgi, Cartan, Fantappiè, Cipolla, Schwerdtfeger and Richter. Attention has been given in only a few cases to the equivalence, or non-equivalence, with other definitions, or to what combinatorial properties of scalar functions were preserved. The casual reader in the field thus gains the impression that a considerable number of essentially distinct extensions of scalar functions to matrices has been achieved.

The principal purposes of this paper are to show that:

(a) All of the definitions except those of Weyr and Cipolla are essentially equivalent.
(b) Weyr's definition is less general than these six, but coincides with them when it is applicable.
(c) Cipolla's definition is more general than these six.
(d) The requirements I–IV of Fantappiè are fulfilled by Cipolla's definition, and V is fulfilled provided $f(z)$ and $g(z)$ are single-valued.

2. The Definitions of Matric Function. Since a polynomial in a complex variable z yields a well-known and satisfactory matric function, it is natural to expect that attempts to extend the concept of a general scalar function would be strongly motivated by properties of matric polynomials. This, we shall see, is the case for most of the eight definitions now to be summarized.

* Fantappiè also imposed a fifth requirement which he needed in order to utilize the theory of linear functionals in arriving at a definition of a matric function.

2.1 *Sylvester, Buchheim*

Sylvester [14], in 1883, gave as definition of a matric function corresponding to the scalar function $f(z)$

$$f(A) = \sum_{j=1}^{n} \prod_{i \neq j} \frac{A - \lambda_i I}{\lambda_j - \lambda_i} f(\lambda_j)$$

where A is a matrix with distinct characteristic roots $\lambda_1, \lambda_2, \cdots, \lambda_n$. This definition, a direct extension of the Lagrange interpolation formula for a polynomial $p(z)$ of degree n, is applicable only when A has distinct characteristic roots.

Buchheim [4], in 1886, generalized Sylvester's definition to the case where the characteristic roots are not necessarily distinct. The generalized interpolation formula of Lagrange

$$p(z) = \sum_{j=1}^{t} \prod_{i \neq j} (z - a_i)^{s_i} \sum_{k=0}^{s_j - 1} \frac{1}{k!} q_j^{(k)}(a_j)(z - a_j)^k, \tag{2.1}$$

where

$$q_j(z) = \frac{p(z)}{\prod_{i \neq j} (z - a_i)^{s_i}},$$

as is well-known, defines uniquely the polynomial of degree $n-1 = \sum_{i=1}^{t} s_i - 1$ or less which, together with its derivatives, assumes the set of prescribed values $p(a_i), p'(a_i), \cdots, p^{(s_i-1)}(a_i)$ at each of the points $z = a_i$, $i = 1, \cdots, t$. If $p(z)$ is a given polynomial of degree $n-1$ or less, then (2.1) is an identity in z, and hence (2.1) will remain valid if z is replaced by any square matrix A.

If, however, A is a square matrix of order n, and the a_i are its distinct characteristic roots λ_i, and the s_i are the multiplicities of those roots, then (2.1) holds with z replaced by *A for any polynomial $p(z)$ whether of degree less than n or not.*

To see this let $p(z)$ be an arbitrary polynomial. Let $c(z) = |A - zI|$. Then by the division algorithm, there exist polynomials $h(z)$ and $r(z)$ such that $p(z) = h(z)\, c(z) + r(z)$ where $r(z)$ is either zero or of degree less than n. It follows that $p(A) = r(A)$. Now the polynomial $p(z) - r(z) = h(z)c(z)$ and also its first $s_j - 1$ derivatives have the value zero at $z = \lambda_j$, for each j. But since the right member of (2.1) is completely determined, for a given A, by the values of the polynomial p and its first $s_j - 1$ derivatives at each λ_j, we have

$$\begin{aligned} p(A) = r(A) &= \sum_{j=1}^{t} \prod_{i \neq j} (A - \lambda_i I)^{s_i} \sum_{k=0}^{s_j - 1} \frac{1}{k!} \frac{d^k}{dz^k} \left[\frac{r(z)}{\prod_{h \neq j} (z - \lambda_h)^{s_h}} \right]_{z = \lambda_j} (A - \lambda_j I)^k \\ &= \sum_{j=1}^{t} \prod_{i \neq j} (A - \lambda_i I)^{s_i} \sum_{k=0}^{s_j - 1} \frac{1}{k!} \frac{d^k}{dz^k} \left[\frac{p(z)}{\prod_{h \neq j} (z - \lambda_h)^{s_h}} \right]_{z = \lambda_j} (A - \lambda_j I)^k \end{aligned} \tag{2.2}$$

which is the right member of (2.1) formed for $p(z)$ and A.

Exactly the same argument could be applied if the s_j represent the multiplicities of the λ_j as roots of the minimum polynomial $m(z)$ of A, using $m(z)$ in place of $c(z)$. Hence (2.2) holds for any polynomial $p(z)$ and any square matrix A with distinct roots $\lambda_1, \cdots, \lambda_t$ of multiplicities in the minimum equation $s_1, \cdots, s_t$.

Now the right member of (2.2), valid for any polynomial function $p(z)$, has meaning if $p(z)$ is any function which is analytic at the repeated zeros of the minimum function and defined at the non-repeated zeros. Hence it is natural to use (2.2) as the definition of any such function of the matrix A. This is Buchheim's definition.

$$(2.3) \qquad f(A) = \sum_{i=1}^{t} \prod_{i \neq j} (A - \lambda_i I)^{s_i} \sum_{k=0}^{s_j-1} \frac{d^k}{dz^k}\left[\frac{f(z)}{\prod_{h \neq j} (z - z_h)^{s_h}}\right]_{z=\lambda_j} (A - \lambda_j I)^k$$

where s_i is the multiplicity of λ_i as a root of the minimum polynomial of A.

2.2 *Power series definition—Weyr*

The power series definition of a matric function probably occurred to a number of mathematicians prior to Sylvester's paper. However, E. Weyr [15], in 1887, appears to have been the first one to give a convergence criterion for a matric power series. The power series definition is a natural extension of polynomial functions of a matrix.

Let $f(z)$ be analytic at $z = z_0$, and

$$(2.4) \qquad f(z) = f(z_0) + f'(z_0)(z - z_0) + \cdots + \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + \cdots .$$

Then one can define the corresponding matric function

$$(2.5) \quad f(A) = f(z_0)I + f'(z_0)(A - z_0 I) + \cdots + \frac{f^{(k)}(z_0)}{k!}(A - z_0 I)^k + \cdots$$

provided the series converges.*

By a translation $y = z - z_0$ of the complex plane, series (2.4) is converted into a Maclaurin series in y with unchanged radius of convergence. At the same time, all of the characteristic roots of $B = A - z_0 I$ are, by the theorem of Frobenius, those of A reduced by z_0, so that the characteristic roots of A are subjected to the same translation. Hence the following theorem of K. Hensel [9] originally stated and proved for a Maclaurin series, is valid for (2.5):

* A sequence of matrices Z_n is said to be convergent if each sequence of corresponding elements is convergent, and the limit matrix is the matrix of the limits of the element sequences. An infinite series of matrices is said to be convergent, if the sequence of partial sums S_n converges.

The power series (2.5) *converges if, and only if, every characteristic root* λ_i *of* A *lies within or on the circle of convergence of* $f(z)$, *and for every root* λ_i *of multiplicity* ν_i *in the minimum polynomial of* A, *the* (ν_i-1)*th derivative* $f^{(\nu_i-1)}(\lambda_i)$ *converges.**

This theorem, an extension of a result of Weyr, settles the question of convergence of matric power series. The condition that all characteristic roots lie within, or on some circle of convergence of $f(z)$ is clearly a more stringent demand than that required by the Buchheim definition, hence the power series definition is less broadly applicable. Further, this defect cannot be overcome by a process of analytic continuation.

To try to sidestep this difficulty Ferrar [7] has employed the following device. Let t be any complex number such that the characteristic roots $t\lambda_i - tz_0$ of $tA - tz_0 I$ are all less in absolute value than the radius of convergence of

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \cdots + \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + \cdots .$$

Then

$$f(tA) = f(z_0)I + f'(z_0)t(A - z_0 I) + \cdots + \frac{f^{(k)}(z_0)}{k!}t^k(A - z_0 I)^k + \cdots$$

will converge, as will be shown in §2.5, to a polynomial in $t(A - z_0 I)$ with coefficients which are polynomials in t and $f^{(j)}(t\lambda_i)$, $j = 1, \cdots, s_i - 1$. This polynomial has a perfectly well-defined value for $t = 1$, which is defined to be $f(A)$.

2.3 *Cartan*

E. Cartan, *circa* 1928, proposed in a letter to G. Giorgi, that the matric function $f(A)$ corresponding to a scalar function $f(z)$ analytic at the characteristic roots $\lambda_1, \cdots, \lambda_n$ of A be defined, by analogy with Cauchy's integral theorem, by

$$f(A) = \frac{1}{2\pi i}\int \frac{f(z)dz}{zI - A} \tag{2.6}$$

where the integral is taken for each element of the matrix $f(z)(zI - A)^{-1}$ around a set of admissible closed paths enclosing each of the distinct characteristic roots of A.

Cartan had in mind that this definition would be more readily extendable to infinite matrices, but apparently did not elaborate his proposed definition further. Giorgi made no observations regarding any relationship of this definition to his own.

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* Hensel's proof of this theorem is given in [11]. Note, however, that the theorem is slightly misstated there.

The Cartan definition is more restrictive than that of Buchheim in that $f(z)$ is required to be analytic at each characteristic root of A.

2.4 *Fantappiè*

L. Fantappiè [6] in 1928, imposed conditions I–IV of §1, and in addition the condition:

V′. *If $f(z, t)$ is analytic in a parameter t, then the elements of $f(A, t)$ depend analytically upon t.*

He then employed the theory of linear functionals to deduce that the elements f_{rs} of $f(A)$ must be given by the sum of the residues of $-(D_{sr}(t)/D(t))f(t)$ at its singularities, where $D_{sr}(t)$ is the cofactor of the r,s element of $A-tI$, and $D(t)=|A-tI|$.

This definition is, as noted by Cipolla, evidently exactly equivalent to that suggested by Cartan, except for the apparent added requirement V′, which, however, follows as a consequence from Cartan's definition. Fantappiè did not discuss the relationship of his definition to those of Giorgi and Cartan which were probably not known to him at the time, nor to those of Weyr and Buchheim.

2.5 *Giorgi*

G. Giorgi [8] in 1928 proposed a definition of a matric function which can be considered to be motivated by a property of polynomial functions $p(z)$, namely that if P is any non-singular matrix then

$$P^{-1}p(A)P = p(P^{-1}AP), \quad \text{or} \quad p(A) = Pp(P^{-1}AP)P^{-1}.$$

Thus if A is reduced to its Jordan normal form C by a similarity transformation, the polynomial $p(C)$ when re-transformed by the inverse similarity transformation will be $p(A)$.

If this principle is postulated for defining a non-algebraic matric function, then the problem of defining $f(A)$ is reduced to that of defining $f(C)$.

Now C is a direct sum $C_1+C_2+\cdots+C_k$ of blocks of the form,

$$C_i = \begin{bmatrix} \lambda_i & 1 & 0 \cdots 0 \\ 0 & \lambda_i & 1 \cdots 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & 0 \cdots \lambda_i \end{bmatrix},$$

and $p(C)=p(C_1)+\cdots+p(C_k)$,* for any polynomial function $p(z)$. To find $p(C_i)$, of order ν_i, we express $p(z)$ in a Taylor expansion about $z=\lambda_i$, getting

$$p(C_i) = p(\lambda_i)I + p'(\lambda_i)(C_i - \lambda_i I) + \cdots + \frac{p^{(\nu_i-1)}(\lambda_i)}{(\nu_i - 1)!}(C_i - \lambda_i I)^{\nu_i-1}$$

since $(C_i-\lambda_i I)^k$ vanishes for $k \geqq \nu_i$. Here I stands for the identity matrix of order ν_i. This yields for the polynomial $p(C_i)$ the matrix

* Note that here the λ_i occurring in different C_i are not necessarily distinct.

$$(2.7)\qquad p(C_i) = \begin{pmatrix} p(\lambda_i) & p'(\lambda_i) & \frac{p''(\lambda_i)}{2!} & \cdots & \frac{p^{(\nu_i-1)}(\lambda_i)}{(\nu_i-1)!} \\ 0 & p(\lambda_i) & p'(\lambda_i) & \cdots & \frac{p^{(\nu_i-2)}(\lambda_i)}{(\nu_i-2)!} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & p'(\lambda_i) \\ 0 & 0 & 0 & \cdots & p(\lambda_i) \end{pmatrix}.$$

Now this definition of a polynomial function of C_i has meaning for a non-algebraic function as well. Hence it is natural to define, with Giorgi, corresponding to a scalar function $f(z)$,

$$(2.8)\qquad f(A) = P[f(C_1) + f(C_2) + \cdots + f(C_k)]P^{-1}$$

where $f(C_i)$ is defined as in (2.7) with f in place of p, $C_1 \dotplus C_2 \dotplus \cdots \dotplus C_k$ being the Jordan normal form of A yielded by the transforming matrix P.

This definition is clearly meaningful under exactly the same circumstances as Buchheim's definition, since the maximum order of the blocks C_i corresponding to a common characteristic root is equal to the multiplicity of that root in the minimum polynomial of A. In particular, if λ_i is a simple root of the minimum polynomial, then all blocks C containing λ_i will be of order one, and Giorgi's (or Buchheim's) definition does not require analyticity of $f(z)$ at $z=\lambda_i$.

The value $f(A)$ given by Giorgi's definition is independent of the matrix P chosen to transform A into canonical form. This may be seen as follows. Let $C=P^{-1}AP$ be a Jordan canonical form of A. By applying a further similarity transformation $R^{-1}CR$, one can bring together as consecutive blocks all those C_i which correspond to a given characteristic root λ. Call these block sets K_j, $(j=1, \cdots, r)$. Then, clearly,

$$f(R^{-1}CR) = R^{-1}f(C)R = R^{-1}[f(K_1) \dotplus f(K_2) \dotplus \cdots \dotplus f(K_r)]R$$

where the characteristic roots of K_i are different from those of K_j, $i \neq j$.

Now each $f(K_i)$ is a polynomial $p_i(K_i)$ in K_i, namely

$$(2.9)\qquad f(\lambda_i)I + f'(\lambda_i)(K_i - \lambda_i I) + \cdots + \frac{f^{(\nu_i-1)}(\lambda_i)}{(\nu_i-1)!}(K_i - \lambda_i I)^{\nu_i-1}$$

where I is an identity matrix of order equal to that of K_i and ν_i is the order of the largest Jordan block in K_i.

By a theorem of M. H. Ingraham [10], there exists a single polynomial $u(z)$, of degree less than the degree of the minimum polynomial of A, such that

$$u(K_i) = p_i(K_i), \qquad i = 1, \cdots, r.$$

Hence

$$f(R^{-1}CR) = R^{-1}f(C)R = R^{-1}(u(K_1) + u(K_2) + \cdots + u(K_r))R$$
$$= R^{-1}(u(C))R = u(R^{-1}CR).$$

Now the polynomial u does not depend on the matrix P chosen to reduce A to Jordan form, nor on the matrix R, since it is determined entirely by the blocks K_i which are independent of R and P. Hence for any P, *i.e.*, for any C and R,

$$f(A) = PRu(R^{-1}CR)R^{-1}P^{-1} = u(PCP^{-1}) = u(A).$$

Since the degree of $u(z)$ is less than that of the minimum polynomial of A, the matric polynomial $u(A)$ is unique.

Thus to each matrix A and to each single-valued function $f(z)$, Giorgi's definition yields a unique value $f(A)$ subject to the restriction already noted on the character of $f(z)$ at the characteristic roots of A.

2.6 *Schwerdtfeger*

H. Schwerdtfeger devoted about half of his doctoral dissertation (Bonn, 1935) to matric functions. He rediscovered Giorgi's definition for the case of distinct roots and showed that it was equivalent to the infinite series definition in the case of an entire function $f(z)$.

From power series considerations he was led in [13] to the following definition for the case of an $f(z)$ analytic at the repeated roots of the minimum polynomial of A. Letting $m(z)$ be the minimum polynomial, he made the partial fraction separation

$$\frac{1}{m(z)} = \sum_{j=1}^{t} \frac{h_j(z)}{(z - \lambda_j)^{s_j}} \tag{2.10}$$

where $\lambda_1, \cdots, \lambda_t$ are the distinct zeros of $m(z)$. He then defined the polynomials $g_j(z)$ by

$$g_j(z) = \frac{h_j(z)m(z)}{(z - \lambda_j)^{s_j}}. \tag{2.11}$$

In terms of the corresponding matrices $G_j = g_j(A)$, the so-called Frobenius covariants of A, he then defined

$$f(A) = \sum_{j=1}^{t} G_j \sum_{k=0}^{s_j-1} \frac{f^{(k)}(\lambda_j)}{k!}(A - \lambda_j I)^k. \tag{2.12}$$

This definition clearly has the same range of applicability as that of Buchheim, $f(z)$ being required to be defined at the characteristic roots of A and to be analytic at the repeated roots of the minimum polynomial. Schwerdtfeger did not relate his definition with any of the definitions given by earlier writers, but did show that the Cartan relation (2.6) was fulfilled by (2.12), and that in the case of distinct characteristic roots (2.12) was equivalent to the Lagrange inter-

polation formula for matrices.

It may be worth noting how (2.12) can be viewed as a generalization of a polynomial function. The matrices G_j, which are polynomials in A, have the following readily verifiable properties:

$$\begin{aligned} \sum_{i=1}^{t} G_i &= I \\ G_j^2 &= G_j, && (j = 1, \cdots, t), \\ G_j(A - \lambda_j I)^k &= 0, && \text{for } k \geqq s_j, \\ G_iG_j &= 0, && i \neq j. \end{aligned} \tag{2.13}$$

From (2.13), for integral $i \geqq 0$,

$$A^i = A^i \sum_{j=1}^{t} G_j = \sum_{j=1}^{t} A^i G_j = \sum_{j=1}^{t} (AG_j)^i.$$

Hence, if $p(z)$ is any polynomial,

$$p(A) = \sum_{j=1}^{t} p(AG_j). \tag{2.14}$$

Now if each $p(AG_j)$ is evaluated from the Taylor expansion of $p(z)$ about $z = \lambda_j$,

$$p(AG_j) = p(A)G_j = G_j \sum_{k=0}^{s_j-1} \frac{p^{(k)}(\lambda_j)}{k!} (A - \lambda_j I)^k. \tag{2.15}$$

This together with (2.14) yields (2.12) for the case of a polynomial $p(A)$, and since this also has meaning where $p(z)$ is non-algebraic it is therefore natural to view (2.12) as a definition of matric function.

For later work it is important to note that if $P^{-1}AP = C$ is in Jordan canonical form, then $P^{-1}G_jP$ is a matrix which has 1's in the diagonal positions corresponding to λ_j, and zeros elsewhere. For, from (2.11)

$$P^{-1}G_jP = P^{-1}h_j(A)P \prod_{i \neq j} (P^{-1}AP - \lambda_i I)^{s_i} = h_j(C) \prod_{i \neq j} (C - \lambda_i I)^{s_i}.$$

Now the right member is a direct sum of blocks all of which are zero except those corresponding to λ_j. From $\sum G_i = I$ follows $\sum P^{-1}G_iP = I$ and hence, each $P^{-1}G_jP$ is of the form stated.

In his thesis Schwerdtfeger used the characteristic function in (2.10), (2.11) and (2.12). The definition of $f(A)$ so obtained would be slightly less general than that of Buchheim since $f(z)$ is required to be analytic at the repeated roots of the *characteristic* polynomial of A.

2.7 *Cipolla*

M. Cipolla [5] in 1932 gave an extension of Giorgi's definition which, for a multiple valued function $v(z)$, yields additional meaningful values for $v(A)$ besides those provided by the Giorgi definition.

As in Giorgi's definition A is first reduced to a Jordan normal form.

$$C = P^{-1}AP = C_1 + C_2 + \cdots + C_k.$$

A value of $v(C)$ is defined by

$$v(C) = v_1(C_1) + v_2(C_2) + \cdots + v_k(C_k)$$

where $v_i(z)$ is any single-valued branch of $v(z)$. The various possible values of $v(A)$ are then given by $v(A) = P^{-1}v(C)P$ where all values of $v(C)$, and all matrices P such that $P^{-1}AP = C$, are used.

If the same branch of $v(z)$ is used throughout the blocks corresponding to a given characteristic root λ_j, then this definition coincides with that of Giorgi, for a single-valued $f(z)$ may be composed by choosing branch $v_j(z)$ in a region enclosing λ_j but no other characteristic root. For such a function $f(z)$ the Giorgi and Cipolla definitions yield the same value, and as was shown earlier this value is expressible as a polynomial in A, and the value $f(A)$ is independent of the choice of P.

The same view of a multiple valued function can be taken in the definitions of Buchheim, Schwerdtfeger, Fantappiè, Cartan, and Richter.

If, however, different branches of $v(z)$ are chosen for two or more blocks C_i corresponding to a common characteristic root, values of $v(C)$ will be obtained which are not provided by Giorgi's definition. Furthermore, in this case, $v(C)$ need no longer be a polynomial in C, hence $v(A)$ need not be a polynomial in A, and the value $v(A)$ obtained from $v(C)$ will no longer be independent of the matrix P. Hence, in this case, the choice of the matrix P provides a further proliferation of the values yielded for $v(A)$.

If the reader is interested in applying Cipolla's definition to find the various square roots of the identity matrix of order 2, he will readily see the greater scope of Cipolla's definition.

2.8 *Richter*

Richter [12], in 1951, treated what he called the general continuous function, F_u, of a matrix, whose principal imposed properties were:

(1) The elements of $F_u(A)$ are continuous functions of the elements of A;

(2) $$F_u(P^{-1}AP) = P^{-1}F_u(A)P.$$

For a matric function corresponding to a scalar function $f(z)$ he resorted to a power series approach to define a functional value for a Jordan block, and ultimately defined

$$f(A) = \sum_{j=1}^{t} R_j \sum_{k=0}^{s_j-1} \frac{1}{k!} f^{(k)}(\lambda_j)(A - \lambda_j I)^k \tag{2.16}$$

where

$$R_j = w_j(A) \prod_{i \neq j} (A - \lambda_i I)^{s_i},$$

and the polynomials $w_j(z)$ are any polynomials satisfying

$$w_j(z)\prod_{i\neq j}(z-\lambda_i)^{s_i}\equiv 1, \qquad \text{mod } (z-\lambda_j)^{s_j},$$

where s_j is the multiplicity of λ_j as a root of the minimum polynomial of A. The matrix $f(A)$ obtained does not depend on the choice of the $w_j(z)$.

The similarity of form of Richter's definition to those of Buchheim and Schwerdtfeger is evident. Richter, however, did not appear to have been aware of any of the earlier work in this field except that of Fantappiè.

It is clear that the sphere of applicability of Richter's definition is precisely that of Schwerdtfeger's.

3. The relationship of the definitions. To facilitate comparison of the definitions, some shorthand symbolism is desirable. Let D_i, $i=1, \cdots, 8$ represent symbolically the various definitions of matric function given in the preceding eight sub-sections. Let us define:

$$D_i = D_j$$

if the definitions D_i and D_j, (a) for each scalar function $f(z)$ are applicable for exactly the same set of matrices and (b) yield the same value, or set of values, for each $f(z)$ and admissible A; further define: $D_i \subset D_j$ if D_j is applicable whenever D_i is, but not conversely, and if for each $f(z)$ and admissible A for which both D_i and D_j are applicable, the values $f(A)$ are the same for both D_i and D_j; the relationship of the various definitions can then be stated in

THEOREM 1. $D_2 \subset D_3 = D_4 \subset D_1 = D_6 = D_8 = D_5 \subset D_7$.

Proof.

In the discussion of the various definitions we have already pointed out the conditions under which each definition was applicable, and have thus established that aspect of Theorem 1, which is concerned with the relative breadths of applicability of the definitions.

It remains, therefore, to establish that the values ascribed to $f(A)$ are the same for any two definitions applicable to $f(z)$ and A. We have already observed this for D_5 and D_7. The balance of the theorem will then be proved, if we show $D_i \subseteq D_5$, $i \neq 5, 7$.

We prove first that $D_2 \subset D_5$. Following essentially Hensel's proof of his theorem, we let, formally,

$$f(A) = \sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(z_0)(A - z_0 I)^i. \tag{3.1}$$

Consider the partial sums ω_k of this series; ω_k is a polynomial in A and hence, if P is a matrix transforming A to Jordan normal form $\omega_k(A) = P(\omega_k(P^{-1}AP))P^{-1}$ with $\omega_k(P^{-1}AP) = \omega_k(C_1) + \omega_k(C_2) + \cdots + \omega_k(C_t)$, where according to subsection 2.5, for $k \geqq$ max of ν_j,

$$\omega_k(C_j) = \begin{bmatrix} \omega_k(\lambda_j) & \omega_k'(\lambda_j) & \cdots & \frac{1}{(\nu_j-1)!}\omega_k^{(\nu_j-1)}(\lambda_j) \\ 0 & \omega_k(\lambda_j) & \cdots & \frac{1}{(\nu_j-2)!}\omega_k^{(\nu_j-2)}(\lambda_j) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \omega_k(\lambda_j) \end{bmatrix}$$

where $\omega_k(z)$ is the corresponding partial sum of $f(z)$. Now it is clear that the right member of (3.1) converges if and only if the series $f(z)$ and each of its derivatives up to order ν_j-1 converge for $z=\lambda_j$, (which incidentally proves Hensel's theorem) and in this case $f(A)=\lim_{k\to\infty}\omega_k=P^{-1}G(P^{-1}AP)P$ where $G(P^{-1}AP)$ here means the value of "$f(C)$" according to Giorgi. Hence $D_2\subset D_5$.

If the argument of the previous paragraph is carried through for the matrix $t(A-z_0I)$, and t is subsequently replaced by 1, it is evident, since the characteristic roots of tA are $t\lambda_i$, that Ferrar's proposed extension of the power series definition is equivalent to Giorgi's definition, except for analyticity requirements.

Having previously shown that $D_3=D_4$, we shall next prove $D_4\subset D_5$. We first note that from the definition of the integral of a matrix, as the matrix of integrals of the elements, it is readily seen that a constant matric factor can be taken inside or outside the integral sign at will. Accordingly let $P^{-1}AP$ be in Jordan normal form. Then

$$\frac{1}{2\pi i}\int \frac{f(z)dz}{zI-A} = \frac{1}{2\pi i}\,P^{-1}\cdot\left\{\int \frac{f(z)dz}{P^{-1}(zI-A)P}\right\}P$$

$$= \frac{1}{2\pi i}\,P^{-1}\left\{\int \frac{f(z)dz}{zI-P^{-1}AP}\right\}P.$$

Now $zI-P^{-1}AP$, considered as a matrix with elements in the field of all rational functions of z with complex coefficients, is non-singular and possesses an inverse. Further this inverse will be a direct sum of the inverses of the matrices $zI-C_i$, where C_i has the meaning of subsection 2.5, and I is the identity matrix of the same order as C_i. It is easily seen that

$$(zI-C_i)^{-1} = \begin{bmatrix} (z-\lambda_i)^{-1} & (z-\lambda_i)^{-2} & \cdots & (z-\lambda_i)^{-\nu_i} \\ 0 & (z-\lambda_i)^{-1} & \cdots & (z-\lambda_i)^{-(\nu_i-1)} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & (z-\lambda_i)^{-1} \end{bmatrix}$$

and hence $(1/2\pi i)\int f(z)(zI-C_i)^{-1}dz$, taken around a set of admissible closed curves each enclosing a distinct characteristic root of A, gives precisely the $f(C_i)$ of Giorgi's definition, $(1/2\pi i)\int(z-\lambda_i)^{-k}f(z)dz$ being zero for each contour enclosing $\lambda_j\neq\lambda_i$, and equal to $(1/(k-1)!)f^{(k-1)}\lambda_i$ for the contour about λ_i.

We next show that D_6 equals D_1. D_6 can be written*

$$f(A) = \sum_{j=1}^{t} \prod_{i \neq j} (A - \lambda_i)^{s_i} \sum_{k=0}^{s_j-1} \frac{f^{(k)}(\lambda_j)}{k!} h_j(A)(A - \lambda_j I)^k,$$

or, expressing the polynomial h_j, of degree $s_j - 1$ or less, as $\sum_{m=0}^{s_j-1} b_m (A - \lambda_j I)^k$,

$$f(A) = \sum_{j=1}^{t} \prod_{i \neq j} (A - \lambda_i I)^{s_i} \sum_{k=0}^{s_j-1} \frac{f^{(k)}(\lambda_j)}{k!} \sum_{m=0}^{s_j-1} b_m (A - \lambda_j I)^{k+m}. \tag{3.2}$$

Since $\prod_{j=1}^{t} (A - \lambda_j I)^{s_j} = 0$, we need only consider those terms in the last sum for which $k + m < s_j$.

Now comparing (3.2) with Buchheim's definition, (2.3),

$$\sum_{j=1}^{t} \prod_{i \neq j} (A - \lambda_i I)^{s_i} \sum_{k=0}^{s_j-1} \frac{1}{k!} \frac{d^k}{dz^k} \frac{f(z)}{\prod_{h \neq j} (z - z_h)} \Bigg]_{z=\lambda_j} (A - \lambda_j I)^k,$$

the equality of the two formulas will be established if we show that the coefficients of $(A - \lambda_j I)^k$ in the second sums are the same in the two formulas: that is, we wish to show that

$$\frac{1}{k!} \frac{d^k}{dz^k} \left(\frac{f(z)}{\psi_j(z)} \right) \Bigg]_{z=\lambda_j} = \sum_{m=0}^{k} \frac{f^{(m)}(\lambda_j)}{m!} b_{k-m} \tag{3.3}$$

where

$$\psi_j(z) = \prod_{i \neq j} (z - \lambda_i)^{s_i}.$$

From (2.10),

$$\frac{1}{m(z)} = \sum_{i=1}^{t} \frac{h_i(z)}{(z - \lambda_i)^{s_i}}$$

follows

$$\frac{1}{\psi_j(z)} = \sum_{i=1}^{t} \frac{h_i(z)(z - \lambda_j)^{s_j}}{(z - \lambda_i)^{s_i}} = \phi_j(z)$$

from which

$$\phi_j^{(m)}(\lambda_j) = h^{(m)}(\lambda_j), \qquad \text{for } m < s_j.$$

Now

$$b_r = \frac{1}{r!} h^{(r)}(\lambda_j) = \frac{1}{r!} \phi^{(r)}(\lambda_j), \qquad \text{for } r < s_j.$$

Hence the right member of (3.3) can be written

* This proof was given by F. H. McGar, Jr. in an unpublished master's thesis, 1951.

$$\frac{1}{k!}\sum_{m=0}^{k}\frac{f^{(m)}(\lambda_j)}{m!}\cdot\frac{k!}{(k-m)!}\phi_j^{(k-m)}(\lambda_j)$$

which by Leibniz's formula can be written

$$\frac{1}{k!}\frac{d^k}{dz^k}\left(\frac{f(z)}{\psi_j(z)}\right)\Bigg]_{z=\lambda_j}$$

which is the left member of (3.3).

To show that $D_8=D_7$ it is only necessary to observe that the definition of the $h_j(z)$ by

$$\frac{1}{m(z)}=\sum_{j=1}^{t}\frac{h_j(z)}{(z-\lambda_j)^{s_j}}$$

follows

$$\sum_{j=1}^{t}h_j(z)\prod_{i\neq j}(z-\lambda_i)^{s_i}=1.$$

In other words the $h_j(z)$ satisfy the congruences

$$h_j(z)\prod_{i\neq j}(z-\lambda_i)^{s_i}\equiv 1, \qquad \mod (z-\lambda_j)^{s_j},$$

and thus can serve as the polynomials $w_j(z)$ in Richter's definition. Richter's formula for $f(A)$ is thus seen to be identical with the Schwerdtfeger definition.

It remains to show that $D_6=D_5$. According to D_6

$$f(A)=\sum_{j=1}^{t}G_j\sum_{k=0}^{s_j-1}\frac{f^{(k)}(\lambda_j)}{k!}(C-\lambda_jI)^k. \tag{3.4}$$

Then

$$P^{-1}f(A)P=\sum_{j=1}^{t}P^{-1}G_jP\sum_{k=0}^{s_j-1}\frac{f^{(k)}(\lambda_j)}{k!}(C-\lambda_jI)^k$$

where $C=C_1+C_2+\cdots+C_r$ is the Jordan normal form of A. Now from subsection 2.6, $P^{-1}G_jP$ has zeros everywhere except in the diagonal positions corresponding to λ_j, where it has 1's. Hence term number m of the right member of (3.4) is a direct sum of Giorgi submatrices

$$\begin{pmatrix} f(\lambda_m) & f'(\lambda_m) & \cdots & \frac{1}{p!}f^{(p)}(\lambda_m) \\ 0 & f(\lambda_m) & \cdots & \frac{1}{(p-1)!}f^{(p-1)}(\lambda_m) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f(\lambda_m) \end{pmatrix}$$

corresponding to the distinct root λ_m. Thus $f(A)$ as given by (3.4) is identical with $f(A)$ as given by Giorgi.

4. The fundamental laws of combination for matric functions. Among the principal writers on matric functions only Fantappiè, Cipolla and Schwerdtfeger concerned themselves with the fundamental question as to whether matric functions, as they defined them, obeyed the elementary rules of combination of scalar functions, although the utility of matric functions depends critically upon these rules. Fantappiè proved that *if* I–IV of section 1 are satisfied, and his additional condition V′ is fulfilled, then *necessarily* the matric function must be of the form he developed. He did not verify, however, that the formula he arrived at thereby necessarily must satisfy I–IV. Cipolla asserted without proof that his definition satisfied Fantappiè's requirements I–IV.

A direct application of Cipolla's definition to the functions k and z makes I and II obvious. The fulfillment of III is also clear, with the precaution that if in $h(z)=f(z)+g(z)$, $f(z)$ and/or $g(z)$ are multiple valued, then the determination used for $h(A)$ is dictated by the determinations used for $f(A)$ and/or $g(A)$. That is, (1) the choice of the branches of $h(z)$ relative to a given block of the Jordan normal form of A must be the sum of the branches of $f(z)$ and $g(z)$ used for that block; and (2) the same transforming matrix P must be used for each of $f(A)$, $g(A)$ and $h(A)$.

The fulfillment of IV is demonstrated as follows.* Let $h(z)=f(z)g(z)$. Then a value of $f(A)g(A)$ is, if the same matrix P is used in computing both $f(A)$ and $g(A)$,

$$\begin{aligned} f(A)g(A) &= P\,|\,f_1(C_1)+f_2(C_2)+\cdots+f_r(C_r)\,| \\ &\quad\cdot|\,g_1(C_1)+g_2(C_2)+\cdots+g_r(C_r)\,|\,P^{-1} \\ &= P\,|\,f_1(C_1)g_1(C_1)+f_2(C_2)g_2(C_2)+\cdots+f_r(C_r)g_r(C_r)\,|\,P^{-1}. \end{aligned}$$

Now a direct computation shows that $f_i(C_i)g_i(C_i)$ is again a triangular matrix with diagonal elements $f_i(\lambda_i)g_i(\lambda_i)$. The element in row r column s above the diagonal is, writing simply $f_i^{(j)}$ for $f_i^{(j)}(\lambda_i)$,

$$\sum_{j=0}^{s-r-1}\frac{1}{j!(s-1-r-j)!}f_i^{(j)}\,g_i^{(s-1-r-j)} = \frac{1}{(s-r-1)!}\sum_{j=0}^{s-r-1}\binom{s-r-1}{j}f_i^{(j)}\,g_i^{(s-1-r-j)},$$

which by Leibniz's formula is $(1/(s-r-1)!)(f_ig_i)^{(s-r-1)}$. This is exactly the element in row r column s of $h(C_i)$, for the choice $f_i(z)g_i(z)$ as branch of $h(z)$ for block C_i. Hence, if the same transforming matrix P is used in determining $h(A)$ for this set of branch choices, then $h(A)=f(A)g(A)$.

We now prove the important

* This proof is given by Ferrar in [7]. A variation of it is found in [13].

THEOREM 2.* *Let A be a square matrix with the distinct characteristic roots $\lambda_1, \cdots, \lambda_t$ of multiplicities in the minimum polynomial of A $\mu_1, \cdots, \mu_t$, respectively. Let $g(z)$ be a function which is single valued at each λ_i and analytic at those λ_i whose $\mu_i > 1$. Let $f(z)$ be a function which is single valued at $z_i = g(\lambda_i)$, $i = 1, \cdots, t$ and is analytic at those z_i corresponding to $\mu_i > 1$. Let $h(z) = f(g(z))$. Then $h(A)$ is uniquely defined by Cipolla's (i.e., Giorgi's) definition and $h(A) = f(g(A))$.*

To compute $f(g(A))$ by Cipolla's definition, one begins by reducing $g(A)$ to Jordan normal form, recalling that under the assumption of single-valuedness the value of $f(g(A))$ will be independent of the matrix chosen to effect that reduction. First let P be a matrix reducing A to Jordan form. Then $P^{-1}f(g(A))P = f(P^{-1}g(A)P)$, where $P^{-1}g(A)P$ is a direct sum of blocks of the form

$$K_{p\lambda} = \begin{pmatrix} g(\lambda_p) & g'(\lambda_p) & \cdots & \dfrac{1}{(\mu-1)!}g^{(\mu-1)}(\lambda_p) \\ 0 & g(\lambda_p) & \cdots & \dfrac{1}{(\mu-2)!}g^{(\mu-2)}(\lambda_p) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & g(\lambda_p) \end{pmatrix}$$

corresponding to the blocks $C_{p\lambda}$ of the Jordan form of A.

Now $P^{-1}g(A)P$ can be reduced to Jordan form by transforming by a matrix R which is a direct sum of matrices $R_{p\lambda}$ each of which transforms the corresponding block $K_{p\lambda}$ into Jordan form. We now form the functional value of each of the resulting Jordan blocks according to Cipolla. By (2.9) of subsection 2.5, for each of the Jordan blocks in $R_{p\lambda}^{-1}K_{p\lambda}R_{p\lambda}$, and hence for their direct sum,

$$f(R_{p\lambda}^{-1}K_{p\lambda}R_{p\lambda}) = \sum_{r=0}^{\mu-1} \frac{1}{r!} f^{(r)}(g(\lambda_p))(R_{p\lambda}^{-1}K_{p\lambda}R_{p\lambda} - g(\lambda_p)I)^r$$

holds, whence

$$R_{p\lambda}^{-1}f(K_{p\lambda})R_{p\lambda} = R_{p\lambda}^{-1}\left\{\sum_{r=0}^{\mu-1} \frac{1}{r!} f^{(r)}(g(\lambda_p))(K_{p\lambda} - g(\lambda_p)I)^r\right\} R_{p\lambda}.$$

Hence

$$f(K_{p\lambda}) = \sum_{r=0}^{\mu-1} \frac{1}{r!} f^{(r)}(g(\lambda_p))(K_{p\lambda} - g(\lambda_p)I)^r. \tag{4.1}$$

We now wish to show that $f(K_{p\lambda})$ as given by (4.1) is the function $f(g(z))$ formed for the block $C_{p\lambda}$ of $P^{-1}AP$. The term $r=0$ in (4.1) gives $f(g(\lambda_p))$ as the

* A rather complicated proof of this theorem is given by Ferrar [7] for the case of convergent power series. In [13] Schwerdtfeger offers a proof of this theorem which appears to be based on one or more unwarranted assumptions or else to omit some crucial details.

diagonal elements of $f(K_{ph})$. We delete this term from the sum and write $K_{ph}-g(\lambda_p)I$ in terms of powers of the matrix

$$Q=\begin{bmatrix}0 & 1 & 0\cdots 0\\ 0 & 0 & 1\cdots 0\\ \cdot & & \quad 1\\ \cdot & & \\ 0 & 0 & 0\cdots 0\end{bmatrix}$$

which has 1's along the first super-diagonal and zeros elsewhere. Clearly Q^i is a matrix having 1's along the ith super-diagonal and zero's elsewhere. Hence, in terms of Q

$$K_{ph}=g(\lambda_p)I+g'(\lambda_p)Q+\frac{1}{2!}g''(\lambda_p)Q^2+\cdots+\frac{1}{(\mu-1)!}g^{(\mu-1)}(\lambda_p)Q^{\mu-1}$$

and (4.1) can be written

$$(4.2)\quad f(K_{ph})-g(\lambda_p)I=\sum_{r=1}^{\mu-1}\frac{f^{(r)}(g(\lambda_p))}{r!}\left[g'(\lambda_p)Q+\cdots+\frac{g^{(\mu-1)}}{(\mu-1)!}(\lambda_p)Q^{\mu-1}\right]^r.$$

This shows that $f(K_{ph})$, being a polynomial in Q, will have all terms equal along a given super-diagonal.

Since all derivatives of f appearing are evaluated at $g(\lambda_p)$ and those of g at λ_p, we shall henceforth dispense with writing the arguments.

We now wish to show that the coefficient of Q^β, $\beta<\mu$, in (4.2) is equal to

$$\frac{d^\beta}{dz^\beta}[f(g(z)]_{z=\lambda_p}.$$

By the multinomial theorem the coefficient of Q^β in (4.2) is, since $Q^\mu=0$,

$$\sum_{r=1}^{\mu-1}\frac{f^{(r)}}{r!}\left\{\sum_{(s)}\frac{r!}{\alpha_1!\alpha_2!\cdots\alpha_\beta!}\prod_{i=1}^{\beta}\left[\frac{g^{(i)}}{i!}\right]^{\alpha_i}\right\},$$

where the inner sum is taken over all non-negative integral solutions (s) of the equations

$$(4.3)\qquad \sum_{i=1}^{\beta}\alpha_i=r,\qquad \sum_{i=1}^{\beta}i\alpha_i=\beta.$$

Now the coefficient of Q^β can be rewritten as

$$\frac{1}{\beta!}\sum_{r=1}^{\beta}f^{(r)}\left\{\sum_{(s)}\frac{\beta!}{\alpha_1!\alpha_2!\cdots\alpha_\beta!}\prod_{i=1}^{\beta}\left[\frac{g^{(i)}}{i!}\right]^{\alpha_i}\right\}.$$

By a theorem due to Faà di Bruno [16], this is precisely

$$\frac{1}{\beta!}\frac{d^\beta}{dz^\beta}[f(g(z))],$$

evaluated of course at $z=\lambda_p$.

This shows that the submatrix of $P^{-1}f(g(A))P$ corresponding to a Jordan block of $P^{-1}AP$ is exactly the same as the corresponding submatrix of $h(P^{-1}AP)$. Hence

$$h(A) = Ph(P^{-1}AP)P^{-1} = f(g(A)).$$

The preceding proof fails if either f or g is not single-valued, since the values of the matric function are then no longer independent of the transforming matrix, nor is the functional value of a K_{ph} expressible as a polynomial in A. The precise conditions under which $h(A)=f(g(A))$ for multiple-valued f and/or g, are not known.

5. Observations on Cipolla's definition.

5.1. The definition given by Cipolla is the most general which has been given for a matric function corresponding to a given scalar function. However, even Cipolla's definition does not give all of the values desired for a matric function in some cases, as has been pointed out by V. Amato [1], G. Andreoli [2] and Richter [12].

It can happen that a function $f(z)$ may not be analytic at a multiple root of the minimum polynomial of A, in which case Cipolla's definition fails, although there may be a perfectly "respectable" value for $f(A)$. Consider, for example, $f(z)=\pm\sqrt{z}$ for the argument

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is clear that Cipolla's definition yields the sole value

$$f(A) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

although evidently

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is also a desired value to ascribe to $f(A)$.

This phenomenon will always occur, for instance, when $f(y)$ is the inverse function of a function $y=g(z)$, analytic at a repeated root of the minimum polynomial of A, for which $g'(\lambda)=0$, since then $f'(g(\lambda))$ is undefined.

Richter has indicated a method by which one may recover such desired values of an inverse function. Andreoli treated the general concept of matric functions from the standpoint of matric power series with matric coefficients.

He was not led to a definition of a matric function corresponding to a given scalar function different from that of Cipolla, but did show that there existed values of a matric function of the category just mentioned.

5.2 *Generalization of Frobenius's theorem.*

A well known and useful theorem in the algebra of matrices is that due to G. Frobenius: *Let A be any square matrix of order n with characteristic roots $\lambda_1, \cdots, \lambda_n$ and $f(z) = g(z)/h(z)$ be any rational function over the complex field, such that $h(A)$ is non-singular. Then the characteristic roots of the matrix $f(A)$ are precisely $f(\lambda_1), f(\lambda_2), \cdots, f(\lambda_n)$.*

From Cipolla's definition, it is obvious that Frobenius's theorem is valid for any function $f(z)$ and for any A to which the definition applies.

5.3 *Functional Values which are polynomials in A.*

In sections 2.5 and 2.6 it was shown that a single valued function $f(z)$, analytic at the repeated roots of the minimum polynomial of a matrix A, always yields a matric value $f(A)$ which is a polynomial in A. This has sometimes been misinterpreted as saying that any function of a matrix is simply a polynomial function. This is not the case, for the polynomial that arises for a given matrix A depends upon the argument matrix, and may be different for different A's. In other words for an indeterminate matrix X, $f(X)$ cannot be expressed as a polynomial in X with constant coefficients.

That $f(A)$ is a polynomial in A is somewhat analogous, for a scalar function of a complex variable considered as a generalization of a real function, to the fact that the value of a function $g(z)$ at an imaginary value $z = a$ can be expressed as a polynomial in a with real coefficients (indeed a polynomial of degree no greater than 1). This, of course, is far from saying that $g(z)$ is a polynomial in z.

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DIGITAL COMPUTERS

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In the field of digital computers great strides have been taken in the past decade. Since the present widespread availability of these machines is having a broad impact on applied mathematics, and since so many of our mathematics graduates are finding employment in computing laboratories attached to industrial and government establishments, it may be of interest to review the present status of this development.

During the war, the tremendous number of ballistic tables needed to describe the flight of the new missiles as they became available made the Army's Ordnance Department and the Navy's Bureau of Ordnance willing customers of the men who came forth with the idea of exploiting Babbage's old (1833) concept of a machine that would compute on the basis of orders given to it before it began its work—a machine that could remember. The Harvard Mark I, the first machine with this power, was developed by the IBM company and Professor Howard Aiken, and worked for the Bureau of Ordnance from the time it began operating in 1944 until after the war. And the Bell Telephone Laboratories Relay Computer gave round-the-clock service at Aberdeen. Toward the end of the war the ENIAC, the first electronic machine, was under development at the University of Pennsylvania for Aberdeen; and this machine, in spite of its limited capacity, achieved revolutionary speeds in the multiplication of two numbers. Its success pointed the way toward the vast changes that were to come.

Although all these machines were being developed by the military primarily to solve the ordinary differential equations of ballistics, they had many other more exciting uses; and there were a few men like John von Neumann, who realized the pressing need for much more powerful machines, capable of handling complex scientific problems, if our atomic efforts were to prosper; and a few others, like Norbert Wiener, who saw the character of the changes they might introduce into industry. This was in 1946. The ENIAC began successful operation in 1947.

At the beginning of 1955, the editors of *Management Methods* published a report on "What Management should know about Electronics for the Office" in which they said:

"The time has come for the businessman to drop the future tense as far as the Automatic Office is concerned. Managements who fail to grasp this new condition—and its implications—may find themselves hopelessly outdistanced by alert competitors before they have time to act.

Here are the facts. . . .

(Some misleading or controversial statements have been omitted)

"Fact I It will take a company not less than 24 months—perhaps longer—to prepare itself for automatic data processing equipment. This does *not* include lead-time for machine delivery. You can reasonably assume a three year target date for getting into electronic operation if you start from scratch to-day.

"Fact II Virtually any firm employing more than 100 clerical workers is ripe for some form of electronic data processing.

"Fact III The price of electronic data processing equipment is consistent with its ability to provide a proper rate of return. Example: a $1,000,000 computer system will pay for itself in three to four years if it replaces only 50 clerical employees.

"Fact IV Not all electronic systems cost $1,000,000. Lower priced, special purpose machines are already in actual use by private firms. New equipment is coming out fast.

"Fact V The fear of obsolescence of present-day equipment is no excuse for failure by management to act. 'ENIAC' . . . is 'obsolete' but still in profitable operation.

"Fact VI Human beings are going to be replaced in staggering numbers by electronic equipment. . . . *The effect of electronic equipment on our economic life is of the same magnitude as the effect of the H-bomb on our military strategy.*"

They go on:

"If businessmen have failed to recognize the actual advent of the Automatic Office, they are not wholly to be blamed. Ample evidence of a general nature has been available for several years, but only recently has specific proof been published. One big barometer could have been read four years ago when almost every one of the large business machine firms committed millions of dollars to electronic development. Companies like Burroughs, IBM, National Cash Register, Remington Rand, and Underwood—all with a tremendous stake in the status quo—made the irrevocable decision that their present products were 'obsolete.' . . . Dozens of the real giants like U. S. Steel and General Electric have completed their long preparation for electronic data processing and have equipment on order or delivered."

The business-machine companies mentioned in this quotation, and others, have all recognized the need of industrial personnel for training in the use of the new computers, and have instituted formal course work to provide this training. For example, Remington Rand has four courses. The first runs for two weeks, and is given about six times a year. It is designed for business executives and members of electronic evaluation committees, and is aimed at providing a general understanding of the techniques for applying general purpose electronic computers to business problems. There are two courses in programming, both running for six weeks, the elementary one given about six times a year, and the advanced one about three times a year. And, finally, there is a course given three times a year, lasting six weeks, designed for operation and maintenance personnel. National Cash Register has a four week basic course in programming, and a six week advanced course. Incidentally it may be worth mentioning that the National Cash Register machine is one of those costing less than a million dollars, and is a general purpose machine.

Probably the most important operational *Fact* among the quotations given above is the first one: It will take a company not less than 24 months—perhaps longer—to prepare itself for automatic data processing equipment. This fact is related to one of the most important characteristics of the great electronic "brains." They are capable of performing remarkable feats, with great speed and accuracy; but they must be told, in the greatest detail, exactly what they are to do. As Claude Shannon has put it, "a digital computer must be instructed in words of one microsyllable." And it is a fact that in a large business operation there is probably no one person who knows in detail and can write down in the most elementary form, all the steps that are taken in a complete business operation. It has proved to be worthwhile in several companies merely to analyze the operation itself, in an attempt to evaluate the desirability of introducing computers. For the analysis of the operation may well disclose undesirable complications and duplications, even though the decision may be reached that data-processing equipment is not needed.

What is the nature of these machines and what types of problems have they been solving?

Mechanical aids to computation go back at least as far as the abacus, but the distinctive thing about these modern machines is that they are automatically sequenced—they can be given instructions *before* they begin to operate, which will enable them to perform each tiny step that they must take in order to complete the problem. With modern electro-mechanical desk computers, as with the abacus, each computational step requires direct human intervention; but with the automatically sequenced machine, the human operator must think through, in fullest detail, the many steps that must be taken, and tell the machine about them on the instruction tape. There are advantages and hazards in this fact. With a desk calculator, the speed of the total operation is subject to human physiological limitations. With an automatically sequenced machine, the speed of computation is vastly increased, but the programming and coding

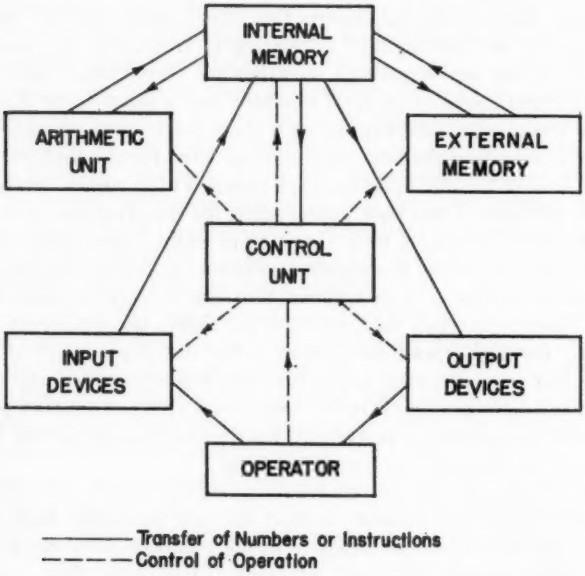
must take into account all the acts of memory and judgment that usually go into a desk calculation.

This means that a code (an instruction tape), once written, must be subjected to a period of "debugging," consisting of trial runs that show up some (hopefully all) of the omissions and mistakes that have been built into the instructions. In recent tests of an IBM machine and a Remington Rand machine on a weather forecasting problem, though there were other considerations that were decisive in selecting the best available machine for the problem, it was discovered that coding for one machine took about 6 man-weeks, while debugging the code took 15 hours of machine time; coding for the other machine took about 17 man-weeks, and debugging took about 50 machine hours. Since one machine hour on these two machines is roughly equivalent to four or five hundred man-weeks on a desk computer, you will see that the debugging operation is non-trivial. One device that has met, at least partially, the problems inherent in correct coding for high speed computers is the development of the library of sub-routines. For example, after a sine function has been coded and run successfully, whenever a sine function is to be computed as part of another problem, the coder splices in the previously tested sub-routine for this operation. This part of the new code then needs no further debugging.

If the disadvantages of extended debugging are great, the advantages of vastly increased speed are greater, so that the new machines have taken their places in our fast moving world. Original applications of these machines were to scientific problems which involved relatively little basic data, but required many thousands, and sometimes many millions of operations on these data. Since little information needed to be fed into the machines, and the answers were short, the original emphasis was on speed within the computing element, with little emphasis on speed in the input and output mechanisms. Business and management problems, on the other hand, ordinarily involve the performance of relatively simple mathematical operations on great masses of data. The answers, too, are sometimes very long. Thus effective utilization of the speed of electronic computers in the solution of problems of this type requires greatly increased speed in input and output mechanisms. In the recent past, there has been a good deal of emphasis on the solution of these problems, and some of the present input and output equipments have attained impressive speed. Many machines are equipped to type out results on an electrically operated typewriter at about 10 characters per second, but output printers operating 40 to 200 times as fast as this are beginning to find widespread use in business practice.

How are these machines put together and how do they operate? Though a full answer to this question would go far beyond the scope of this paper, some of the basic notions can be given. The parts of a computer are usually referred to as organs; and this designation reflects the similarity of their functions to the functions of certain organs of a human being. The block diagram of the machine may be given as:

BLOCK DIAGRAM OF AN AUTOMATIC DIGITAL COMPUTER



The operator puts problems on the computer through the input devices, causes it to commence operation through a signal to the control unit, and receives the results from the output devices. The input is typically punched paper tape, punched cards, magnetic tape or magnetic wire, although photographic film has also been tried. Whichever of these devices is used records the instructions and the data of the problem, and feeds them into the memory. The control interprets the instructions to the other parts of the machine and causes the desired arithmetical operations to be performed.

For most machines the preferred arithmetic is binary, as contrasted with our usual decimal system. This means that all numbers are represented by using only the two digits 0 and 1. Thus 1 1 0 1 represents

$$1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2 + 1$$

The reason for this nearly universal use of some form of binary representation is that the typical equipment used in these machines has two states. Thus punched cards and tape are always either punched or not punched in each position, and magnetic tapes, wire and drums distinguish most reliably a pulse from a no-pulse. Likewise, electronic circuits are most easily built with two stable states.

Often the input and output are handled in decimal form, while internally the computer deals with numbers in binary representation, the conversion being made by the computer internally.

The operations that one of these "giant brains" can perform are essentially only the elementary operations of addition, subtraction, multiplication, and sometimes division, as well as one or two other orders like the comparison of two numbers to decide which is greater. This instruction makes it possible to instruct the computer to do one thing if, say, the first number is greater, and a different thing if the second number is greater. Addition is handled either by our usual method, from right to left, "carrying" the 1 when a sum gives ten or more (called in the machine serial addition), or by another method, parallel addition, which adds all the digits simultaneously, as in a desk calculator, reserving for later consideration the inclusion of the "carried" digits. Very fast machine methods have been devised for handling addition in this way.

Some of the characteristics of digital machines now in operation are:

1. They will perform multiplication at speeds sometimes as fast as 30 millionths of a second.
2. They have internal memories sometimes of several thousand words (either numbers or instructions), and external memories many times as large.
3. They are able to follow instructions in succession, and set up automatically the proper connections between machine parts.
4. If at any point in the calculation there are two or more alternative courses for the subsequent operation, they will select and perform the correct one according to specified conditions contained in their instructions.

Something more should be said about the memory devices that are in use, since the speed of computation is importantly affected by this organ, and the kind of problem a machine can handle depends significantly on the size and speed of the memory.

The earliest electronic machines (after the ENIAC) used mercury delay lines as the memory element. A little later magnetic drums were perfected; and then the research on electrostatic storage, which had been initiated very early, was finally successful and produced several varieties of vacuum tube storage. What do we mean by these terms?

A mercury delay line is a form of acoustic storage that consists of a number of tanks containing mercury and equipped with a quartz crystal at each end. Electrical impulses representing digital information are impressed on one of the crystals and are converted by it into acoustical energy in the mercury. The pulses then travel through the tank as sound waves, and are reconverted into electrical pulses by the crystal at the other end. After being corrected for loss of amplitude and definition, the pulses are fed back into the input of the tank. The information can then be recirculated continuously through the tank without loss of synchronism or definition, thus forming a memory device storing pulses the

number of which is determined by the length of the pulse and the length of time required for an acoustic disturbance to travel the length of the tank. The first computer with a mercury delay line memory to operate successfully was the EDSAC, designed and constructed at the Mathematical Laboratory of the University of Cambridge, England. It was completed in 1949. It was inspired by the EDVAC, a machine that was then under construction for Aberdeen at the University of Pennsylvania.

The characteristic multiplication speed of American machines using this type of memory is 2 or 3 thousandths of a second.

When these delay line machines began to operate they increased the multiplication speed by a factor of about one hundred over the speeds available with the older machines that had used telephone relays as the memory element (the multiplication speed of the ENIAC was comparable with that of the delay line machines, but the ENIAC'S limitation was its tiny memory—only 20 words).

Magnetic drum memories which were developed at about the same time as the delay line memories have proved the most reliable and the easiest to put into successful operation. They have been widely adopted by the manufacturers of the more moderately priced machines that are smaller in size and are intended for relatively small businesses. Most of these machines have attained a multiplication speed about one-tenth that of the delay line machines.

A magnetic drum is a cylinder four inches to a foot in diameter and two inches to three feet in length which is coated with a thin layer of ferromagnetic material. Magnetic heads serving, in turn, for both reading and writing, are placed along the drum, often with a clearance of one mil. These heads, between about ten and two hundred in number are generally mounted in groups, with about ten to the inch in the axial direction. The drum must be very accurately balanced so that, in rotating at a speed of perhaps 3600 rpm or faster, there is very little variation in the head-to-drum spacing.

The longest and most trying effort went into the development of the electrostatic tube memory, which makes use of the secondary-emission characteristic and high resistivity of the cathode ray tube phosphor. For a long time the hopes of American machine designers rested in a special tube that was being developed by the Radio Corporation of America, the Selectron tube. But F. C. Williams in Manchester, England, succeeded in building a memory using ordinary cathode ray tubes before the Selectron was put into operation. The Williams principles have been adapted to the design of the American machines, and this is the type of electrostatic tube machine most widely used in America. The speed of multiplication of these machines is about ten times that of the delay line machines.

Although the details of operation of this type of memory are complicated, the basic idea is simple enough. On a single cathode ray tube, a beam is made to store small positive or negative charges, resulting from secondary emission or a spray of secondary electrons, at the points of an array sometimes as dense as 32×32 on the screen of the tube. One cathode ray tube is used for each binary

digit of a word, the digits being recorded in corresponding positions on the faces of the several tubes.

These various types of memory are often used in combination. For example, one of the large commercially available machines has a fast internal memory of over 20,000 digits of electrostatic storage, an auxiliary magnetic drum storage of over 80,000 digits, and an external, magnetic tape memory of over 8,000,000 digits without changing tapes. A second large computer which is on the market has an internal memory of nearly 600,000 digits in magnetic drum storage and up to 150,000 digits of magnetic core storage. This magnetic core storage is the newest and most promising very fast memory element. It is made up of tiny magnetic rings strung together in a matrix. Within the next few years there are bound to be important machines built with this type of storage. The size and speed of this element make its use consistent with that of transistors, and this is undoubtedly a direction in which we may expect developments.

What is the story of the transition I have suggested from the immediate post-war years when the new machines were being developed by the Army and Navy primarily for the solution of ballistic problems to the present wide-spread interest in industry in the use of electronic computers for many facets of the problems of management? Clearly the present situation is a concomitant of large investments in the design and construction of high speed computers by some of the wealthiest companies operating in America today.

In 1946, and continuously until 1951 when the first UNIVAC operated successfully, the big machines were always believed to be just 18 months in the future. After World War II the concern to exploit the potentialities of digital computers for the benefit of federal departments not engaged extensively in military research was expressed first by the Census Bureau which secured the services of the Bureau of Standards as a contracting and monitoring agent in the purchase of a digital computer which was to be used in compiling the 1950 census. This was the UNIVAC that began operating in 1951. However, the staff of the Bureau of Standards, after repeated failures to secure delivery from its various contractors—and there were several other federal agencies for which the Bureau was serving as contracting agent—itself embarked on the construction, with a minimum of development, of an acoustic delay line computer, using the results that had been obtained by the other groups working in the field. This was the first successful delay line computer to operate successfully in America. It built upon the pioneering work in logical design of John von Neumann and his Institute for Advanced Study group as did virtually all other computers not only in the United States but in the world. It benefited by the successes and the difficulties of the EDVAC group at the University of Pennsylvania, which had been working for some time on a delay line computer for Aberdeen. It used much of the current know-how about input and output equipment, and it developed some very clever ideas all its own. But most important, it operated, at a time when many important activities in the United States were sorely in need of a

machine of its speed. This was the SEAC—the Standards Eastern Automatic Computer (as the name suggests, the West-coast branch of the Bureau was also in the business, and ultimately, but much later, had a successful machine in the SWAC—the Standards Western Automatic Computer). It was the SEAC that did the early automatic computations that were done for the Census Bureau, although much of the later work could wait for the UNIVAC (also a delay line computer). After the UNIVAC had been put into successful operation, one by one, other machines began to operate; and some machines that had been in operation, but on classified military problems, were adapted for the commercial market. Remington Rand and IBM embarked on extensive construction and sales programs; many of the small companies (a number of these had been established by imaginative young engineers who wished to enter the competitive market with machines of moderate price) were bought up by other large business-machine companies. Thus the automatically sequenced digital computer came of age as an element in industrial management.

Why has speed of multiplication been emphasized? Since the early machines were designed primarily for the solution of scientific problems, the essential time element to be considered was the time to make the computations that occurred within the machine; and by far the decisive one of these was multiplication. On one fast machine, for example, addition may take 36 microseconds, while multiplication takes 500. Thus a reasonable estimate of the speed of operation of a machine used primarily for scientific computations is given by the multiplication speed. For business uses, like inventory control, this observation is no longer true. Here, the number of items of input is enormous, compared with the computations performed on the items. Thus, input and output equipment becomes most important. We are now capitalizing on the great speeds that have been developed for the internal computing element in such esoteric applications as Russian translations and weather forecasting; and we are developing fast end-organs to take care of business needs.

In the early post war days, this whole development was financed almost exclusively by the government. More recently private capital has taken over responsibility for the specifically business uses, and has further developed the speed and versatility of the internal computing elements.

To give some idea of the scope of the present applications of automatically sequenced computers, a brief list is given of some of the types of scientific problems that have been and are being solved on large scale computers, and some of the categories of business applications that have been developed for these machines. Among scientific problems, solutions of the partial differential equations of hydrodynamics (whose solution in three dimensions involves millions of computations) play an important part; for these equations occur in weather prediction and in explosion theory as well as in various other aspects of wave problems. A sample of problems that have been attacked by these machines includes the computation of the so-called structure factors in crystallography (of great importance in metallurgy) analysis and design of optical systems by ray-

tracing techniques, analysis of emission and reflection of radiant energy from the surfaces of radiant heated enclosures, study of the vibration of aircraft wings, the computation of air flow past a delta wing or rocket fin, analysis of frictional gas flow in jet motors, study of shielding problems in the design of atomic reactors. One of the early IBM super-calculators spent a considerable portion of the time it devoted to scientific problems computing the positions of the five outer planets—Jupiter, Saturn, Neptune, Uranus, and Pluto—at 40-day intervals for a period of 407 years, from 1653 to 2060. The operation involved over 5,000,000 multiplications and divisions, and more than 7,000,000 separate additions and subtractions of large numbers. When this machine sold its time, the charge was $300 per hour. But it was available without charge for problems of general interest to science originating in university or similar research.

Among business problems, studies of inventory control and related questions have received considerable attention, and some machines have been designed specially to handle this problem. It involves simple operations, performed over and over again, and a very simple and inexpensive machine can make vast improvements in operation. General purpose machines have also been programmed for this problem as well as for sales analysis, billing, preparation of payroll on a completely automatic basis, taking account of the great variety of individual deductions and adjustments. At the Bureau of Standards, the SEAC has recently been used to determine the low bidder on government contracts. In general, this analysis is extremely complex. Features that raise complications are: simultaneous purchase for several depots, involving different freight costs from factories to depot and consequent price fluctuation within a given lot; price variation depending on size of purchase from a particular manufacturer, with minimum acceptable and maximum available quantity varying from one manufacturer to another; the necessity for determining the actual lowest bidder on the total transaction exactly and not approximately because of the requirements set by law. The mathematical analysis used in the solution of this problem was based on the method of linear programming developed in connection with the study of logistics problems by the Defense department. This is an illustration of mathematical methods that are emerging, and in fact must be developed if the new machines are to be exploited to anything like their full potential. A new kind of applied mathematics is coming of age.

General purpose computers are in operation in England, Canada, Belgium, Switzerland, Holland, Sweden, Australia, France, Germany, Norway, and Japan, as well as other countries about which we have less information. There are surely over seventy in operation in the United States, with the number constantly rising. The development goes on, with the expanded use of these machines providing one element of concern as the completely automatic factory and automatic office raise problems for labor and management alike.

A NOTE ON THE CLASSICAL GROUPS

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The purpose of this paper is to set forth some topological properties of and relations between certain classical groups. While much of the material included here may be known to a few, the main interest of this paper lies in the simplicity of the proofs of some important, though obscured, results. The groups considered are the full linear group of $n \times n$ matrices over the complex numbers, denoted by $GL(n, c)$, and some of its subgroups: $GL(n, r)$, the subset having real coefficients; $SL(n, c)$, the subset of $GL(n, r)$ with determinant unity; $U(n)$, the set of unitary matrices; $SU(n)$, the unimodular group; $O(n)$, the real orthogonal group; $SO(n)$, the rotation group; and $GL(n, r)^+$, the elements of $GL(n, r)$ having positive determinants. The symbolic notions and definitions of the groups mentioned above are the same as used in [1].

1. Decompositions as topological spaces. It has been shown in [1] that the underlying space of $GL(n, c)$ is the topological product of the space of the n-dimensional unitary group and that of the positive definite hermitian matrices. A second decomposition of this space gives $GL(n, c)$ as the topological product of the non-zero complex numbers and the space of $SL(n, c)$, the commutator subgroup of the algebraic group $GL(n, c)$ and a subgroup of the topological group $GL(n, c)$. Consider the matrix $A \in GL(n, c)$ and define the matrix $P = (p_{ij})$ such that $p_{11} = |A|$, $p_{ii} = 1$ for $i \neq 1$, and $p_{ij} = 0$, $i \neq j$. Also define $B = (b_{ij})$ by letting $b_{1j} = a_{1j}/|A|$ and $b_{ij} = a_{ij}$ for $i \neq 1$. Clearly $PB = A$ and $|B| = 1$. Since P and B are continuous functions of A, the decomposition is continuous. The intersection of all matrices of type P and all matrices of type B contains only the identity, and the decomposition is therefore unique. Since multiplication of matrices is continuous, the identification is a homeomorphism. The set of all matrices of type P is homeomorphic to the non-zero complex numbers.

Simple considerations of the above decompositions of $GL(n, c)$ will yield two decompositions of the underlying topological space of $GL(n, r)$. First, if $\sigma \in GL(n, r)$, then $\sigma = \tau\alpha$, where τ is unitary, α a positive definite hermitian matrix, and $\sigma = \bar{\sigma}$. Since the decomposition of σ, as an element of $GL(n, c)$, is unique; $\tau\alpha = \bar{\tau}\bar{\alpha}$ implies $\tau = \bar{\tau}$ and $\alpha = \bar{\alpha}$; therefore τ is orthogonal and α is positive definite symmetric. Thus, it is proved that the underlying topological space of $GL(n, r)$ is the topological product of that of $O(n)$ and the space of positive definite symmetric matrices. Secondly, let $A \in GL(n, r)$, then $A = PB$ where P and B are defined as above, and $B \in SL(n, c)$. Similarly, $A = \overline{A}$ and the uniqueness of the second decomposition in the preceding paragraph implies that P is real, $B \in SL(n, r)$, and that the underlying space of $GL(n, r)$ is the topological product of the non-zero reals and the underlying space of $SL(n, r)$. A third decomposition, due to the Gram-Schmidt orthogonalization process, gives $GL(n, r)$ as the product of the space of $O(n)$ and that of the triangular matrices with $a_{ii} > 0$.

It follows easily that the space of $GL(n, r)^+$, the component of the identity in $GL(n, r)$, can be decomposed into (1) the product of $SO(n)$, the n-dimensional rotation group, and the set of positive definite symmetric matrices (2) the product of the positive reals and $SL(n, r)$, and (3) the space of $SO(n)$ and the triangular matrices with $a_{ii}>0$.

Let τ be an element of $SL(n, r)$, then $\tau=\sigma\alpha$; σ is orthogonal and α is positive definite symmetric, since τ is also an element of $GL(n, r)$. Since the determinant of τ equals one, σ and α are of determinant one and the uniqueness of the decomposition in $GL(n, r)$ proves that $SL(n, r)$ is the topological product of the underlying space of $SO(n)$ and that of the positive definite symmetric matrices of determinant one. A similar argument shows that this space, $SL(n, r)$, is also the topological product of $SO(n)$ and the space of all triangular matrices having determinant one and main diagonal elements positive. That $SL(n, c)$ can be decomposed into the product of $SU(n)$ and the space of positive definite hermitian matrices of determinant one can be proved easily using the first decomposition of $GL(n, c)$. Moreover, $U(n)$ is shown to be the product of $SU(n)$ and the complex numbers of absolute value one in [1].

2. Connectivity. It is well known that many of the underlying spaces are connected [1]. But by proving $SU(n)$ is connected by first principles, the connectivity of the others comes easily through a consideration of their decompositions and an elementary property of projections. Any $S\in SU(n)$ can be written as UBU^{-1} with U unitary and B a diagonal matrix having $b_{jj}=e^{i\theta_j}$. Define B_t to be the diagonal matrix formed from B by letting $b_{tjj}=e^{it\theta_j}$ for $0\leqq t\leqq 1$. It is clear that $UB_tU^{-1}=f(t)$ is a curve from B to 1. *And $SU(n)$ is therefore connected.* Since $U(n)=T^1\times SU(n)$, and T^1 is homeomorphic to the unit circle, *$U(n)$ is connected.* The set of positive definite hermitian matrices is homeomorphic to a Euclidean space [1]. This fact together with the connectivity of $U(n)$ and the first decomposition of $GL(n, c)$ proves that *$GL(n, c)$ is connected.* Since the projection function of a product onto one of its factors is continuous, *$SL(n, c)$ is a connected set* (consider the second decomposition of $GL(n, c)$). *$GL(n, r)^+$ is a connected set,* because it is the component of 1 in $GL(n, r)$. $SO(n)$ is the image of this connected set under the projection function on the first decomposition of $GL(n, r)^+$, and *$SO(n)$ is therefore connected.* Similarly $SL(n, r)$, being the image of this function on the second decomposition of $GL(n, r)^+$, *is a connected set.*

3. Deformation retracts. A space X is a deformation retract of Y relative to X if there exists a homotopy f of Y into Y such that (1) $f(0, y)=y$, (2) $f(1, y)\subset X$ for all $y\in Y$, and (3) $f(t, x)=x$ for all $x\in X$.

The following considerations will show that $SO(n)$, $O(n)$ and $U(n)$ are deformation retracts of $GL(n, r)^+$, $GL(n, r)$, and $GL(n, c)$ respectively. The mapping

$$f(t, \tau) = \sigma(t1 + (1 - t)\alpha) \qquad 0 \leqq t \leqq 1$$

is clearly a homotopy of $GL(n, r)^+$, $GL(n, r)$ and $GL(n, c)$, where σ is an element of $SO(n)$, $O(n)$, and $U(n)$ respectively and α is a positive definite symmetric matrix if $\tau \in GL(n, r)^+$ and if $\tau \in GL(n, r)$, and a positive definite hermitian matrix if $\tau \in GL(n, c)$ (see first decompositions of these groups in section 1). Clearly $(t1+(1-t)\alpha)$ is symmetric (hermitian) according as α is symmetric (hermitian), since each set of matrices is closed with respect to scalar multiplication and addition. Moreover, if $\{\lambda_{ii}\}$ are eigenvalues of α, then $\{t+(1-t)\lambda_{ii}\}$ are eigenvalues of $t1+(1-t)\alpha$. Moreover for any t these eigenvalues are positive, if the λ_{ii} are positive. Finally if $\tau \in GL(n, r)^+$, $GL(n, r)$ and $GL(n, c)$, respectively, then $f(0, \tau)=\tau$, $f(1, \tau)=\sigma$, and $f(t, \sigma)=\sigma$ for all t.

Secondly, consider the mapping

$$f(t, A) = \frac{1}{(\sqrt[n]{|A|})^t} A, \qquad 0 \leqq t \leqq 1,$$

on each of the following sets: $GL(n, r)^+$, the set of positive definite symmetric matrices, the set of positive definite hermitian matrices, and the set of real triangular matrices $A=(a_{ij})$ with $a_{ii}>0$. It is clearly a homotopy of these sets. Moreover, using this homotopy one proves easily that $SL(n, r)$, the set of positive definite symmetric matrices of determinant one, the set of positive definite hermitian matrices of determinant one and the set of real triangular matrices $A=(a_{ij})$, $a_{ii}>0$ and $|A|=1$ are respectively deformation retracts of the above named sets.

Finally, let τ be an element of $SL(n, r)$, then by the second decomposition of this space, $\tau=\sigma\alpha$, where σ is an element of $SO(n)$ and $\alpha=(a_{ij})$, a triangular matrix having $a_{ii}>0$ and $|\alpha|=1$. Define $\alpha_t=(b_{ij})$, such that $b_{ii}=a_{ii}^t$, and $b_{ij}=ta_{ij}$ for $i \neq j$ and $0 \leqq t \leqq 1$. The mapping

$$f(t, \tau) = \sigma\alpha_t$$

is clearly a homotopy of $SL(n, r)$. Moreover, it is easily seen that $f(0, \tau)=\sigma$, $f(1, \tau)=\tau$ and $f(t, \sigma)=\sigma$ for each t. Therefore $SO(n)$ is a deformation retract of $SL(n, r)$.

4. Homotopy types. It is well known that if Y_0 is a deformation retract of Y relative to Y_0, then Y_0 and Y are of the same homotopy type [3]. From the preceding considerations it follows that the following sets of spaces are of the same homotopy type:

(1) $U(n)$ and $GL(n, c)$
(2) $O(n)$ and $GL(n, r)$
(3) $SO(n)$, $GL(n, r)^+$ and $SL(n, r)$
(4) The set of positive definite symmetric (hermitian) matrices and the set of positive definite symmetric (hermitian) matrices of determinant one.
(5) The set of real triangular matrices with positive diagonal terms and its subset of matrices with determinant one.

It might be observed at this point that the set of positive definite symmetric matrices and the set of real triangular matrices with positive diagonal terms are homeomorphic, and hence are of the same homotopy type. For each of these sets is homeomorphic to $R^{n(n+1)/2}$. One notes that each is homeomorphic to $R^{n(n-1)/2}\times R^n(+)$, where $R^n(+)$ is that octant of R^n with all coordinates positive; this set is clearly homeomorphic to R^n.

Finally, since homeomorphic sets are of the same homotopy type and since this last relation is one of equivalence, the sets of symmetric matrices in (4) and the spaces in (5) are of the same homotopy type.

5. Further considerations of $SL(n, c)$, $U(n)$ and $SU(n)$. This paper does not establish $SU(n)$ as a deformation retract of $SL(n, c)$. But one observes that their n-th homotopy groups are isomorphic. The set of positive definite hermitian matrices of determinant one is homeomorphic to R^{n^2-1}. Therefore $SL(n, c) = R^{n^2-1}\times SU(n)$. It has been shown elsewhere [2], that if X_1, X_2, and X_3 are topological spaces and if $X_1 = X_2\times X_3$, then $\pi_n(X_1) = \pi_n(X_2)\times\pi_n(X_3)$, where $\pi_n(X_i)$ are n-th homotopy groups of X_i. Also since any map into R^k is homotopic to the identity, $\pi_n(R^k) = I$ for all n and any k, and $\pi_m[SL(n, c)] = \pi_m[SU(n)]$ for all m. Since the m-th homotopy groups of the non-zero complex numbers and of the complex numbers of absolute value one contain only the identity for $m>1$, it follows that $\pi_m[GL(n, c)] = \pi_m[SL(n, c)] = \pi_m[SU(n)] = \pi_m[U(n)]$, for all $m>1$. Moreover, $\pi_m[GL(n, c)] = \pi_m[U(n)]$ for all m.

6. Further consideration of $GL(n, r)$, $O(n)$, $GL(n, r)^+$, $SL(n, r)$, $SO(n)$. The first two of these groups are not connected sets [1], but they are homogeneous spaces with $GL(n, r)^+$ and $SO(n)$ as components of their identity elements respectively. In [2, 114], one observes that $\pi_m[O(n)] = \pi_m(SO(n))$. Therefore these five groups have isomorphic m-th homotopy groups for $m\geqq 1$, since the first two groups are of the same homotopy type and the last three are of the same homotopy type. It is known that m-th homotopy groups are invariants of the same homotopy type [2].

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MATHEMATICAL NOTES

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AN INEQUALITY FOR POSITIVE DEFINITE MATRICES

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Making use of Hölder's inequality and an identity involving multiple integrals, R. Bellman [1] recently showed that, for any hermitian positive definite matrices A, B and any number α such that $0<\alpha<1$,

$$|\alpha A + (1-\alpha)B| \geqq |A|^{\alpha}|B|^{1-\alpha}, \tag{1}$$

where $|X|$ denotes the determinant of the matrix X. Our object is to present an alternative proof of (1) and to generalize this inequality.

We note, in the first place, that

$$\alpha\lambda + 1 - \alpha \geqq \lambda^{\alpha} \qquad (0 < \alpha < 1; \lambda > 0), \tag{2}$$

with equality if and only if $\lambda=1$. This is proved by considering $\alpha\lambda+1-\alpha-\lambda^{\alpha}$ as a function of λ and using the first mean value theorem of the differential calculus.

Next, we recall that two hermitian positive definite matrices A and B (say of order n) can be reduced simultaneously to diagonal form. More precisely, there exists a non-singular matrix P such that

$$\overline{P}'AP = \Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_n), \qquad \overline{P}'BP = I,$$

where $\lambda_1, \cdots, \lambda_n$ are the (necessarily real) roots of the equation $|A-\lambda B|=0$. Moreover, since $\overline{P}'AP$ is again positive definite, it follows that $\lambda_1, \cdots, \lambda_n>0$. We now have

$$\alpha A + (1-\alpha)B = \overline{P}'^{-1}(\alpha\Lambda + (1-\alpha)I)P^{-1},$$

and therefore

$$|\alpha A + (1-\alpha)B| = |\overline{P}'|^{-1}\cdot|P|^{-1}\cdot|\alpha\Lambda + (1-\alpha)I| = |B|\cdot|\alpha\Lambda + (1-\alpha)I|$$

$$= |B|\cdot\prod_{r=1}^{n}(\alpha\lambda_r + 1 - \alpha).$$

Hence, by (2),

$$|\alpha A + (1-\alpha)B| \geqq |B|\cdot\prod_{r=1}^{n}\lambda_r^{\alpha}, \tag{3}$$

and so

$$|\alpha A + (1-\alpha)B| \geqq |B| \cdot |\Lambda|^{\alpha} = |B| \cdot |\overline{P}'AP|^{\alpha} = |B|\,(|\overline{P}'| \cdot |P|)^{\alpha}|A|^{\alpha} = |A|^{\alpha}|B|^{1-\alpha}.$$

This establishes (1). Further, it is clear that there is equality in (1) if and only if there is equality in (3), and that is the case precisely if $\lambda_1 = \cdots = \lambda_n = 1$. Then $\overline{P}'AP = I$ and consequently $A = B$. Thus there is equality in (1) if and only if $A = B$.

It is now easy to deduce the following generalization:

THEOREM. *Let $\alpha_1, \cdots, \alpha_k$ be positive numbers such that $\alpha_1 + \cdots + \alpha_k = 1$, and let $A_1, \cdots, A_k$ be hermitian positive definite matrices. Then*

$$|\alpha_1 A_1 + \cdots + \alpha_k A_k| \geqq |A_1|^{\alpha_1} \cdots |A_k|^{\alpha_k}, \tag{4}$$

with equality if and only if

$$A_1 = \cdots = A_k. \tag{5}$$

We prove this theorem by induction with respect to k. For $k = 2$ it has been demonstrated above. Assume, then, that it holds for $k-1$, where $k \geqq 3$. Since

$$\alpha_1 A_1 + \cdots + \alpha_k A_k = (1-\alpha_k)\left(\frac{\alpha_1}{1-\alpha_k}A_1 + \cdots + \frac{\alpha_{k-1}}{1-\alpha_k}A_{k-1}\right) + \alpha_k A_k$$

and since $\alpha_1/(1-\alpha_k)A_1 + \cdots + \alpha_{k-1}/(1-\alpha_k)A_{k-1}$ is positive definite, it follows by the case $k = 2$ that

$$|\alpha_1 A_1 + \cdots + \alpha_k A_k| \geqq \left|\frac{\alpha_1}{1-\alpha_k}A_1 + \cdots + \frac{\alpha_{k-1}}{1-\alpha_k}A_{k-1}\right|^{1-\alpha_k} |A_k|^{\alpha_k}, \tag{6}$$

with equality if and only if

$$\alpha_1 A_1 + \cdots + \alpha_{k-1}A_{k-1} = (1-\alpha_k)A_k. \tag{7}$$

Moreover,

$$\frac{\alpha_1}{1-\alpha_k} + \cdots + \frac{\alpha_{k-1}}{1-\alpha_k} = 1,$$

and so, by the induction hypothesis,

$$\left|\frac{\alpha_1}{1-\alpha_k}A_1 + \cdots + \frac{\alpha_{k-1}}{1-\alpha_k}A_{k-1}\right| \geqq |A_1|^{\alpha_1/(1-\alpha_k)} \cdots |A_{k-1}|^{\alpha_{k-1}/(1-\alpha_k)}, \tag{8}$$

with equality if and only if

$$A_1 = \cdots = A_{k-1}. \tag{9}$$

From (6) and (8) we now at once infer (4). Also, there is equality in (4) if and

only if there are equalities in both (6) and (8), *i.e.*, if and only if both (7) and (9) are satisfied. But (7) and (9) are together equivalent to (5), and the proof is therefore complete.

It may be pointed out in conclusion that the theorem can also be derived from the generalized inequality of the arithmetic and geometric means and the relation (due to Minkowski)

$$|A_1 + \cdots + A_k|^{1/n} \geqq |A_1|^{1/n} + \cdots + |A_k|^{1/n},$$

valid for any hermitian positive definite matrices $A_1, \cdots, A_k$ of order n, with equality occurring if and only if every pair of A's is linearly dependent.

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CONVEX REGIONS IN PROJECTIVE N-SPACE

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We consider the following conditions on a set K in real projective n-space.

(1) Each point P of K is contained in an n-simplex region which is contained in K.

(2) Any two points P, Q of K are joined by a line segment $\overline{PQ}$ which is contained in K.

(3) Any two points P, Q of K are joined uniquely by a line segment which is contained in K.

(4) There exists an $(n-1)$-dimensional hyperplane of the projective n-space which does not meet the set K.

Condition (1) simply states that K is an open set; when $n=2$, a 2-simplex region is called a triangular region in [1]. A convex region is usually defined [1] as any set satisfying conditions (1), (2), and (4). We wish to prove here that conditions (1) and (3) also characterize the convex regions of a real projective n-space, $n \geqq 1$. Obviously (2) and (4) imply (3), and (3) implies (2). All that remains to be demonstrated is that (1) and (3) imply (4). For the case $n=1$, to deny (4) forces K to be the whole space which is impossible.

For $n=2$, we suppose that every line of the projective plane meets a set K satisfying conditions (1) and (3). Any line λ of the plane would then separate K into two disjoint sets K_1 and K_2 excluding the points of $K \cap \lambda$. K_1 is defined as the set of points which may be joined to some fixed point of K not on λ by segments in K which do not meet λ. K_2 is defined as the set of points of K which are not in K_1 or λ. Clearly K_1 and K_2 each separately satisfy conditions (1) and (4).

To verify that K_1 and K_2 each also satisfy (2) and (3) we first prove that any three non-collinear points P, Q, and R of K are vertices of a triangular region in K. If the segments in K determined by P, Q, and R together with the end-points bound a triangular region, the region is clearly in K. On the other hand, if we suppose the segments in K determined by P, Q, and R together

with the end-points do not bound a triangular region, then we take a point L of $\overline{QR}$ in K and consider the lines LQ' as Q' traverses segment $\overline{PQ}$ in K. We call R' the intersection of LQ' with PR. Let Q'' be the point Q' on segment $\overline{PQ}$ at which the segments $\overline{PQ'}$, $\overline{Q'R'}$, $\overline{R'P}$ in K together with the end-points first occur as not bounding a triangular region in K as Q' traverses $\overline{PQ}$. The points P, Q'', R'' are the vertices of a triangular region in K for which the segment $\overline{R''Q''}$ on its boundary contains a point not in K and the segment $\overline{Q'R''}$ not on its boundary lies entirely in K. Now the lines LQ' where Q' is on $\overline{PQ}$ between P and Q'' meet the triangular region $PQ''R''$ in segments $\overline{Q'R'}$ in K with segments $\overline{R'Q'}$ containing points not in K. Since each point of the segment $\overline{Q'R''}$ in K is contained in a triangular region in K and $\overline{Q'R''}$ with its end-points is compact so that it is covered by a finite collection of the triangular regions in K, we can find a line LQ' for which every point of the segment $\overline{R'Q'}$ as well as $\overline{Q'R'}$, Q', R' would be in K contrary to condition (3).

Now the axiom of Pasch clearly holds for the boundary of any triangular region so that any line not on a vertex meeting one side will meet exactly one of the other sides. When we apply this result to any triangle in K determined by the fixed point used to define K_1 and K_2 and any other two points of K, we see that (3) is true for both K_1 and K_2 and also that any segment in K joining a point of K_1 to a point of K_2 will meet λ.

Let S be any point on λ but not in K. We suppose every line of the pencil on S meets K. As T traverses some line μ not on S, the lines TS are all the lines of the pencil. TS can never meet both K_1 and K_2. There exists on μ a point $T_1 \neq T_0 = \lambda \cap \mu$ such that T_1 is the last T for which TS meets K_1 or is the first T for which TS meets K_2 as T traverses the entire line μ in the sense from K_2 through T_0 to K_1. If T_1S meets K_1 in a point V, then from condition (1) a whole interval of points T on μ about T_1 gives ST meeting K_1 contrary to T_1 being the last such T. A similar contradiction is obtained if T_1S meets K_2. Therefore our supposition concerning the pencil on S is false and (4) is true.

For $n>2$, we proceed in a similar manner using λ as an $(n-1)$-dimensional hyperplane, S as an $(n-2)$-dimensional hyperplane in λ which we know exists not meeting K by assuming our proposition already proved true for $(n-1)$-dimensional real spaces. We take μ as any line which does not meet S, and to complete the proof we consider as before the pencil of $(n-1)$-dimensional hyperplanes ST as T traverses μ. Thus by induction (4) follows from (1) and (3) for all natural numbers n.

From the above result we may state that for $n>1$ it is impossible to find in projective n-space two disjoint sets K and K' satisfying (1) and (3) with K' closure also satisfying (3) for which the union of K and the closure of K' is the whole space. Also any set satisfying (1) and (3) must be homeomorphic to an open convex set in euclidean n-space.

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A NOTE ON EULER TRANSFORMS

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If λ is a positive number and $\phi(t)$ is a non-constant function which is monotone non-decreasing for $0 \leqq t < \infty$, then the function

$$f(x) = \int_0^\infty (x + t)^{-\lambda} d\phi(t), \qquad 0 < x < \infty,$$

will be called the Euler λ-transform of ϕ (*cf.* [1]). The class of functions which can be represented as Euler λ-transforms will be denoted by E_λ. Hartman and Wintner [1] have shown that if λ and μ are two positive numbers such that $\lambda < \mu$, then $E_\lambda \subset E_\mu$. The following result is in the reverse direction.

THEOREM. *If $\lambda > 1$ and $f(x)$ can be written as the Euler λ-transform of a function $\phi(t)$ which is everywhere differentiable in $(0, \infty)$, such that its derivative $\phi'(t)$ satisfies the following conditions*

(i) *$\phi'(t)$ is monotone non-decreasing for $0 < t < \infty$,*

(ii) *$\phi'(t) = O(t^{\lambda-1})$ as $t \to \infty$,*

then $f(x)$ is in $E_{\lambda-1}$.

Proof. Since $f(x)$ is in E_λ, we have

$$f(x) = \int_0^\infty (x + t)^{-\lambda} d\phi(t) = \int_0^\infty (x + t)^{-\lambda} \phi'(t) dt.$$

If we integrate by parts, and make use of condition (ii) we obtain

$$f(x) = [1/(\lambda - 1)]\left[x^{-\lambda+1}\phi'(0) + \int_0^\infty (x + t)^{-\lambda+1} d\phi'(t)\right].$$

Let $\epsilon(t)$ denote the function defined by

$$\epsilon(0) = 0, \qquad \epsilon(t) = 1 \qquad \text{for } t \neq 0,$$

and let

$$\theta(t) = [1/(\lambda - 1)][\phi'(0)\epsilon(t) + \phi'(t)].$$

Then

$$x^{-\lambda+1} = \int_0^\infty (x + t)^{-\lambda+1} d\epsilon(t)$$

and

$$f(x) = \int_0^\infty (x + t)^{-\lambda+1} d\theta(t). \tag{1}$$

It remains to be shown that $\theta(t)$ is non-constant and non-decreasing. If

$\phi'(t)$ is identically equal to some constant a, then a cannot be negative or 0 since $\phi(t)$ is non-constant and non-decreasing. Thus $\alpha>0$ and $\theta(t)=[a/(\lambda-1)][\epsilon(t)+1]$ which is non-decreasing and not constant. If $\phi'(t)$ is not constant we have, for $t\neq 0$,

$$\begin{aligned}\theta(t) &= [\phi'(0)+\phi'(t)]/(\lambda-1)\\ &= \theta(0)+\phi'(t)/(\lambda-1)\end{aligned}$$

which is non-constant. The fact that $\phi'(t)$ is non-decreasing guarantees that $\theta(t)$ is non-decreasing. Thus $f(x)$ is the Euler $\lambda-1$ transform of $\theta(t)$.

Reference

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THE LIMIT OF A CERTAIN INTEGRAL CONTAINING A PARAMETER*

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The purpose of this note is to establish the following estimate:

$$(1)\qquad \begin{cases} I_n \equiv \dfrac{2}{\pi n}\displaystyle\int_0^{\pi/2}\left|\frac{\sin(2n+1)\theta}{2\sin\theta}\cot\theta-\frac{(2n+1)\cos(2n+1)\theta}{2\sin\theta}\right|\sin 2\theta\,d\theta \\ \quad=\dfrac{8}{\pi^2}+O\left(\dfrac{\log n}{n}\right), \qquad (n\to\infty). \end{cases}$$

This integral, with $\theta=\frac{1}{2}\phi$, is found in Rabson [3] where it occurs as $1/n$ times the n-th Lebesgue constant for the convergence of Fourier series on the quaternions of norm one. He shows [3, (5.5)] that

$$(2)\qquad \frac{4}{\pi}+\epsilon \geqq I_n \geqq \frac{2^{1/2}}{\pi}-\epsilon, \qquad n\geqq N_\epsilon$$

for each $\epsilon>0$, all that is needed for his purposes.

The derivation of (1) is along different lines from Rabson's proof of (2) and is based on (3) and (4) below:

$$(3)\qquad \int_0^{\pi/2}\left|\frac{\sin(2n+1)\theta}{\sin\theta}\right|\cos^2\theta\,d\theta = O(\log n);$$

$$(4)\qquad \int_0^{\pi/2}\cos\theta\,|\cos(2n+1)\theta|\,d\theta=\frac{2}{\pi}+O\left(\frac{1}{n}\right).$$

Equation (3) follows from the well-known estimate of the Lebesgue constants arising in the convergence theory of Fourier series, since $0\leqq\cos^2\theta\leqq 1$. As a matter of fact, a much more precise result is obtainable for the integral

* This note was written while the author was the recipient of a Research Corporation grant.

in (3) simply by replacing $\cos^2\theta$ by $1-\sin^2\theta$, separating into two integrals, using the known asymptotic formulas for the Lebesgue constants and the resulting analog of (4). But this is not needed here.

Equation (4) is a special case of a result which, except for the remainder term, is due to Fejér [1]. The remainder term and a somewhat more general result are found in [2, (2.6)].

Now replace $\sin 2\theta$ by $2\sin\theta\cos\theta$, getting

$$I_n = \frac{2}{\pi n}\int_0^{\pi/2}\left|\frac{\sin(2n+1)\theta}{\sin\theta}\cos^2\theta-(2n+1)\cos\theta\cos(2n+1)\theta\right|d\theta$$

$$= \frac{2(2n+1)}{\pi n}\int_0^{\pi/2}\cos\theta\,|\cos(2n+1)\theta|\,d\theta + O\left(\frac{\log n}{n}\right)$$

$$= \frac{8}{\pi^2}+O\left(\frac{1}{n}\right)+O\left(\frac{\log n}{n}\right) = \frac{8}{\pi^2}+O\left(\frac{\log n}{n}\right).$$

Remark. Rabson needs (2) only when n is an integer. The proof of (1), however, is valid for arbitrary, not necessarily integer-valued, growth of n.

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A SHORT PROOF OF A CLASSICAL THEOREM IN THE THEORY OF FOURIER INTEGRALS

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We give here a short self-contained proof of an important theorem of Pringsheim concerning the Fourier integral formula (Math. Ann. vol. 68, 1910 and vol. 71, 1912). We have been unable to find such a proof in the literature.

The theorem in question is the following:

Let f be a complex-valued function of bounded variation on the line $-\infty<t<\infty$, and suppose that $f(t)\to 0$ as $|t|\to\infty$. Then

$$(1)\qquad \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ixy}dy\int_{-\infty}^{\infty}f(t)e^{iyt}dt = \frac{f(x+0)+f(x-0)}{2},\qquad -\infty<x<\infty,$$

where the inner integral is convergent except possibly at $y=0$, and the outer integral is to be interpreted as

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$$\lim_{\epsilon\to 0, R\to\infty}\left(\int_{-R}^{-\epsilon}+\int_{\epsilon}^{R}\right).$$

Our proof makes frequent use of the following lemma which has a certain interest of its own:

LEMMA. *Let f be of bounded variation on the interval $a\leqq t\leqq b$, and let g be continuous on the same interval. Then*

(a).

$$\left|\int_a^b fgdt\right| \leqq V_f(a,b)\sup_{a\leqq\xi,\eta\leqq b}\left|\int_\xi^\eta gdt\right| \quad +\inf_{a\leqq x\leqq b}\left|f(x)\int_a^b gdt\right|$$

$$\leqq \left\{V_f(a,b)+\inf_{a\leqq t\leqq b}|f(t)|\right\}\sup_{a\leqq\xi,\eta\leqq b}\left|\int_\xi^\eta gdt\right|,$$

where $V_f(x, y)$ denotes the total variation of f on the interval $x\leqq t\leqq y$.

(b). *If f is real and vanishes or changes sign in the interval $a\leqq t\leqq b$, then the inequality* (1) *reduces to*

$$\left|\int_a^b fgdt\right| \leqq V_f(a,b)\sup_{a\leqq\xi,\eta\leqq b}\left|\int_\xi^\eta gdt\right|.$$

(The result (b) is not needed here and is given only for the sake of completeness).

The proof of (a) is quite simple. Suppose that f vanishes at one of the end-points of the interval $a\leqq t\leqq b$, say at $t=a$. Integrating by parts, we find that

$$\left|\int_a^b fgdt\right| = \left|\int_a^b\left\{\int_t^b g(u)du\right\}df(t)\right| \leqq V_f(a,b)\sup_{a\leqq\xi,\eta\leqq b}\left|\int_\xi^\eta gdt\right|.$$

Since $V_f(x, y)$ is an additive interval function, this inequality remains true if f vanishes at *any* point of the interval $a\leqq t\leqq b$. The general result follows from the obvious inequality

$$\left|\int_a^b fgdt\right| \leqq \left|f(c)\int_a^b gdt\right| + \left|\int_a^b\{f(t)-f(c)\}g(t)dt\right|.$$

In proving (b), we can, because of (a), restrict ourselves to the case where $f(t)$ is bounded away from zero but changes sign. There is then a finite subinterval at whose end-points $f(t)$ has opposite signs. Bisecting this interval, we get two new intervals of which one and only one has the same property. Continuing in the usual way, we arrive at a nested set of closed intervals whose lengths tend to zero and at whose end-points $f(t)$ has opposite signs. There is a point ξ common to all these intervals. If ξ is a right-hand or left-hand end-point, respectively, of all but a finite number of these intervals, then $f(\xi)$ has sign opposite to $f(\xi-0)$ or $f(\xi+0)$. Otherwise, $f(\xi-0)$ and $f(\xi+0)$ have opposite signs. In either

case, the required result follows from (a) upon redefining f to be zero at ξ. Indeed, this does not alter the integral of fg, and the total variation of the redefined f does not exceed that of f.

The following rather well known properties of

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$$

will be needed in the sequel:

(P1). $$\int_0^\infty \frac{\sin \alpha t}{t}\,dt = \pi/2 \qquad \text{for} \quad \alpha > 0;$$

(P2). (P3). $$\left|\int_\xi^\eta \frac{\sin \alpha t}{t}\,dt\right| \begin{cases} \leqq \operatorname{Si}(\pi) & \text{for} \quad \xi\eta \geqq 0,\ \alpha > 0; \\ \leqq 2/(\alpha\xi) & \text{for} \quad 0 < \xi < \eta = \infty,\ \alpha > 0. \end{cases}$$

Note that the property (P2) is a consequence of the known inequalities $0 \leqq \operatorname{Si}(x) \leqq \operatorname{Si}(\pi)$ for $x \geqq 0$, the oddness of $\operatorname{Si}(x)$, and the fact that the integral in (P2) is equal to $\operatorname{Si}(\alpha\eta) - \operatorname{Si}(\alpha\xi)$.

Now to the proof of the theorem.

First off, the inner integral in (1) converges uniformly for $|y| \geqq \epsilon > 0$ as can be seen by applying the lemma to

$$\int_S^T f(t)e^{iyt}dt$$

as $S,\ T \to \infty\ (\to -\infty)$ and noting that

$$\left|\int_\xi^\eta e^{iyt}dt\right| \leqq 2/|y|$$

for $y \neq 0$ and $S \leqq \xi,\ \eta \leqq T$. Consequently,

$$\begin{aligned} J_{\epsilon,R}(t) &\equiv \frac{1}{2\pi}\left(\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R}\right) e^{-ixy}dy \int_{-\infty}^{\infty} f(t)e^{iyt}dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)dt\left(\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R}\right) e^{iy(t-x)}dy \\ &= I_R(x) - I_\epsilon(x), \end{aligned}$$

where

$$I_a(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+t)\,\frac{\sin \alpha t}{t}\,dt.$$

If we split $I_\epsilon(x)$ into integrals over the intervals $(-\infty, -T]$, $[-T, T]$, and $[T, \infty)$, apply the lemma and (P2) to the first and third of these and use the inequality $|\sin \epsilon t| \leqq \epsilon|t|$ in the remaining one, we see that $I_\epsilon(x) \to 0$ as $\epsilon \to 0$.

We introduce the notation

$$h(t) \equiv h_x(t) \begin{cases} = 0, & t = 0, \\ = f(x+t) + f(x-t) - \{f(x+0) + f(x-0)\}, & t \neq 0. \end{cases}$$

In view of (P1), we then find that

$$\pi\left\{I_R(x) - \frac{f(x+0) + f(x-0)}{2}\right\} = \int_0^\infty h(t) \frac{\sin Rt}{t}\, dt = \int_0^\delta + \int_\delta^\infty .$$

By the lemma and (P2), $\left|\int_0^\delta\right| \leqq \mathrm{Si}(\pi) V_h(0, \delta)$. Since $h(t)$ and, therefore, $V_h(0, t)$ are continuous at $t=0$, it follows that $\left|\int_0^\delta\right|$ can be made arbitrarily small by taking δ small enough. On the other hand, the lemma and (P3) give $\left|\int_\delta^\infty\right| \leqq 2V_h(\delta, \infty)/(R\delta)$ which, for fixed δ, tends to zero as $R \to \infty$.

Remark. It is clear that the above argument applies equally well to show that $J_{x,R}(x) \to f(x)$ uniformly in any proper subinterval of an interval of continuity of f.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

THE IRRATIONALITY OF $\sqrt{2}$

EDWIN HALFAR, University of Nebraska

The following proof has some novelty in that it does not use the usual assumption that the fraction n/m which is to represent $\sqrt{2}$ is such that n and m have no common factors.

Suppose $\sqrt{2} = (n/m)$, where n and m are positive integers. Then $n > m$, and there is an integer $p > 0$ such that $n = m + p$, and $2m^2 = m^2 + 2pm + p^2$. This implies $m > p$. Consequently, for some integer $a > 0$, $m = p + a$, $n = 2p + a$ and $2(p+a)^2 = (2p+a)^2$. The last equality implies $a^2 = 2p^2$ so that the entire process may be repeated indefinitely giving $n > m > a > p > \cdots$, but since every non-null set of positive integers has a smallest element, this is a contradiction and $\sqrt{2}$ is irrational.

ON DISSIMILAR SOLUTIONS OF A DIFFERENTIAL EQUATION

C. H. MURPHY, Ballistic Research Laboratories

A common complaint of undergraduate students in Differential Equations classes is that the major problem is "to put their solutions in the form of the textbook's answer." This problem of reconciling two dissimilar solutions can frequently lead to quite laborious algebraic manipulations. An interesting example of a similar difficulty is provided by a consideration of two apparently conflicting treatments of Van der Pol's equation [1], [2].

By a suitable choice of scale factors Van der Pol's equation may be written in the convenient form:

$$\ddot{y} + y = \mu(1 - y^2)\dot{y}. \tag{1}$$

The problem is to find an approximation to the periodic solution of this equation which reduces to a cosine solution of the linear equation for $\mu = 0$. (This solution is a stable limit cycle and is quite important in non-linear mechanics.) Both authors used a perturbation method which makes use of the small parameter μ. If the solution presented by [1] is corrected for two algebraic errors, it can be written in the form:

$$y = 2\cos \omega t + \frac{\mu}{4}(3\sin \omega t - \sin 3\omega t) + \cdots, \tag{2}$$

$$\omega = 1 - \frac{\mu^2}{16} + \cdots.$$

By a simple transformation McLachlan's solution may be written in a similar form:

$$y = 2\cos \omega t + \frac{\mu}{4}(-\sin \omega t - \sin 3\omega t) + \cdots, \tag{3}$$

$$\omega = 1 - \frac{\mu^2}{16} + \cdots.$$

An examination of these solutions shows that both are periodic and reduce to $2\cos t$ for $\mu = 0$. We see that if squares and higher powers of μ are neglected both have the same value of $y(0)$, but differ in their value of $\dot{y}(0)$. This conflict seems to be irreducible. The solution is, as it happens all too frequently, quite simple. It is clear that a shift in the origin of the time axis is allowable. We, therefore, shift this origin in (2) a small amount by replacing ωt by $\omega t + a\mu$. If squares and higher powers of μ are neglected, we have

$$y = 2\cos \omega t + \frac{\mu}{4}[(3 - 8a)\sin \omega t - \sin 3\omega t]. \tag{4}$$

Hence we see that the choice $a = \frac{1}{2}$ reduces (2) to (3) and that the coefficient of

sin ωt in the *linear* approximation is completely arbitrary. Thus we see, as was pointed out by the referee, there actually is a family of approximations which approach the solution $y=2\cos t$ as μ approaches zero.

References

1. Minorsky, N., Introduction to Non-Linear Mechanics, Edwards Brothers, Ann Arbor, 1947, pp. 158–159.

2. McLachlan, N., Ordinary Non-Linear Differential Equations in Engineering and Physical Sciences, Clarendon Press, Oxford, 1950, pp. 41–46.

INTEGRALS OF INVERSE FUNCTIONS

F. D. Parker, Clarkson College

If $y=f(x)$ and its inverse $x=g(y)$ are single-valued and continuously differentiable, the area under the curve $y=f(x)$ and between the ordinates $x=a$ and $x=b$ may be calculated by using the inverse function. In particular

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)dy.$$

This leads to an interesting formula for indefinite integrals

$$\int^x f(x)dx = xf(x) - \int^{f(x)} g(y)dy.$$

For the undergraduate student, this formula has two uses. First, it gives a method of formal integration for the logarithmic function, the inverse circular and hyperbolic functions, and many others. For example,

$$\int \ln x dx = x\ln x - \int^{\ln x} e^y dy$$

$$= x\ln x - x.$$

Second, it will sometimes determine whether an integral is an elementary function or not. For example, $\int(\ln x)^{1/2}dx$ is not elementary, since $\int e^{y^2}dy$ is not.

The method of deriving this formula suggests a comparison between the "ring" and "shell" methods of finding a volume of revolution, and the two methods of calculating the first moment of area. These lead to the result

$$\int^x f^2(x)dx = xf^2(x) - 2\int^{f(x)} yg(y)dy.$$

This appears to be less useful, but should suggest to the student the two methods of calculating moments of inertia, which lead to the result

$$\int^x f^3(x)dx = xf^3(x) - 3\int^{f(x)} y^2g(y)dy.$$

This suggests the generalization

$$\int^{x} f^{n}(x)dx = xf^{n}(x) - n\int^{f(x)} y^{n-1}g(y)dy.$$

This relation is indeed valid and provides at the undergraduate level an interesting problem in differentiation.

The form of these results suggests that they are closely connected to the formulas obtained by integration by parts. In fact, the last relation may be quickly obtained by making the substitution $x=g(y)$ in the left hand side, then integrating by parts taking $u=y^{n}$ and $dv=g'(y)dy$.

THE FUNDAMENTAL THEOREM OF THE CALCULUS

LOUIS BRAND, University of Cincinnati

Both proof and content of the fundamental theorem of the integral calculus are foreshadowed by the

FUNDAMENTAL THEOREM OF THE SUM CALCULUS. *If $F(n)$ is a function of the integral variable n having $f(n)$ as difference, then*

$$\sum_{n=a}^{b} f(n) = F(b+1) - F(a). \tag{1}$$

The proof is immediate; for the sum in (1) is

$$\begin{aligned}\sum_{n=a}^{b} f(n) &= F(a+1) - F(a) + F(a+2) - F(a+1) + \cdots + F(b+1) - F(b)\\ &= F(b+1) - F(a).\end{aligned}$$

This theorem is readily and effectively illustrated by forming a short table of *anti-differences*. Thus to verify the brief table

$f(n)$	$F(n) = \Delta^{-1}f(n)$
r^{n}	$\dfrac{r^{n}}{r-1}\ (r \neq 1)$
$n^{(k)}$	$\dfrac{n^{(k+1)}}{k+1}$
$\cos n\alpha$	$\dfrac{\sin (n-\frac{1}{2}\alpha)}{2\sin \frac{1}{2}\alpha}$

we need only show that $\Delta F(n)=f(n)$. We can now compute sums such as

$$\Sigma ar^{n}, \qquad \Sigma n^{2} = \Sigma(n^{(2)} + n^{(1)}), \qquad \Sigma \cos n\alpha,$$

between integral limits by use of (1).

A direct analogue of (1) is the

FUNDAMENTAL THEOREM OF THE INTEGRAL CALCULUS. *If $f(x)$ is integrable in the interval (a, b) and $F(x)$ is a function having $f(x)$ as derivative, then*

$$\int_a^b f(x)dx = F(b) - F(a). \tag{2}$$

The proof again depends upon "telescopic" cancellation. Divide (a, b) into subintervals δ_i by the intermediate points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

and let $\delta = \max \delta_i$; then

$$\int_a^b f(x)dx = \lim_{\delta \to 0} \sum_{i=1}^{n} f(t_i)\delta_i, \qquad x_{i-1} \leqq t_i \leqq x_i. \tag{3}$$

Since $F(x)$ is differentiable it is also continuous in (a, b); and from the mean value theorem

$$F(x_i) - F(x_{i-1}) = f(\xi_i)\delta_i, \qquad x_{i-1} \leqq \xi_i \leqq x_i. \tag{4}$$

Since the points t_i in (3) are at our disposal we may take $t_i = \xi_i$. Then the sum in (3) becomes

$$F(x_1) - F(x_0) + F(x_2) - F(x_1) + \cdots + F(x_n) - F(x_{n-1}),$$

that is, $F(b) - F(a)$. Since this is true for any subdivision of (a, b), it must hold as $\delta \to 0$.

The analogy between the theorems is enhanced by writing them in the form:

$$\sum_{n=a}^{b} f(n)\,\Delta n = \Delta^{-1}f(n)\Big|_a^{b+1}; \qquad \int_a^b f(x)dx = D^{-1}f(x)\Big|_a^b.$$

The requirement in the fundamental theorem that $f(x)$ be integrable is essential, for the derivative of a function, $f(x) = F'(x)$, is not necessarily integrable. For example the function

$$F(x) = x^2 \sin \frac{1}{x^2} \; (x \neq 0), \qquad F(0) = 0$$

has the derivative

$$F'(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} \; (x \neq 0), \qquad F'(0) = 0;$$

but since $F'(x)$ is unbounded in any neighborhood of the origin, $F'(x)$ is not integrable over $0 \leqq x \leqq 1$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1171. *Proposed by N. D. Brenner, University of Pennsylvania*

(1) Is a number of the form 2^p-1, where p is a prime, ever divisible by another number of the same form?

(2) Are two such numbers necessarily relatively prime?

E 1172. *Proposed by D. R. Morrison, Tulane University*

Show, for every positive integer n, that of the following sums of binomial coefficients two are equal and the third differs from them by 1; that is, they partition their sum, 2^n, into three parts which are as nearly equal as three integers whose sum is 2^n can be.

$$S_1 = \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots$$

$$S_2 = \binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \cdots$$

$$S_3 = \binom{n}{2} + \binom{n}{5} + \binom{n}{8} + \cdots$$

E 1173. *Proposed by R. A. Laird, New Orleans, La.*

The sum of the lengths of five equal contiguous chords inscribed in a given circular arc is 5280 feet. The length of the long chord of the given arc is 5208 feet. Find, to the nearest inch, the length of the given arc.

E 1174. *Proposed by M. D. Marcus and B. N. Moyls, University of British Columbia*

Let A be an n-square Hermitian matrix whose characteristic roots are the diagonal elements A_{ii}, $i=1, \cdots, n$. Prove that A is diagonal.

E 1175. *Proposed by G. A. Yanosik, New York University*

(1) Three mutually tangent spheres, with radii $r_1<r_2<r_3$, rest upon a horizontal plane. Find the radius R of the largest sphere which will slip through the space between the three given spheres.

(2) Given four mutually tangent spheres of slightly different radii $r_1<r_2<r_3<r_4$, find the radius R of the largest fifth sphere which will fit in the space which is more or less bounded by the four given spheres.

SOLUTIONS

Bounds on a Solution of a Special Riccati Equation

E 1113 [1954, 259 and 714]. *Proposed by Peter Treuenfels, Ballistic Research Laboratory, Aberdeen Proving Ground*

Let $y(x)$ denote that solution of the differential equation $dy/dx=x^2+y^2$ which passes through the origin. Show that $y(1)<23/53$.

II. *Note by Aaron Herschfeld, Polytechnic Institute of Brooklyn.* The value $y(1)=0.350168+$ given in the discussion of this problem is incorrect. From the 1948 *Tables of Bessel Functions of Fractional Order*, vol. 1, put out by the National Bureau of Standards, one finds

$$\begin{aligned} y(1) &= J_{3/4}(0.5)/J_{-1/4}(0.5) \\ &= 0.37110,55199/1.05959,95935 \\ &= 0.35023,18443. \end{aligned}$$

It is perhaps also interesting to note that the simple solution

$$y = xJ_{3/4}(x^2/2)/J_{-1/4}(x^2/2)$$

has its first infinity at the least positive zero of $J_{-1/4}(x^2/2)$, which is easily found from the *NBS* tables. Since $J_{-1/4}(u)$ vanishes between $u=2.00$ and $u=2.01$, a little inverse interpolation gives the zero at $u=2.00629,97$, or $x=2.00314,73$, where $y(x)=\infty$.

Polygonal Path Covering a Square Lattice

E 1123 [1954, 423; 1955, 124]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Given a square $N\times N$ point lattice, show that it is possible to draw a polygonal path passing through all the N^2 lattice points and consisting of $2N-2$ segments. Can it be done with less than $2N-2$ segments?

II. *Addendum by John Selfridge, U.C.L.A.* We answer the question at the end of the problem in the negative.

Let there be R rows and S columns which have none of the given segments lying on them. The $R\times S$ point lattice formed by these has $2R+2S-4$ boundary points if R and S are each greater than 1. Each oblique line covers at most 2 of these boundary points. Thus in the polygonal covering there are at least $R+S-2$ oblique segments, $N-R$ horizontal segments, and $N-S$ vertical segments, or at least $2N-2$ segments in all.

If R or S is 0 or 1 there are at least $2N-1$ segments.

Inradius of a Mixtilinear Triangle

E 1141 [1954, 711]. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Find the radius of the circle inscribed in the mixtilinear triangle formed by the two legs of a given right triangle ABC and the semi-circumference described externally upon the hypotenuse AB.

I. *Solution by the Proposer.* Let ω be the center and ρ the radius of the circle inscribed in the mixtilinear triangle. The semi-circumference and its reflection in the line AB is the circumcircle of triangle ABC. Extend CA and CB their own lengths to A' and B' and draw $A'B'$. The circumcircle of triangle ABC is the nine-point circle of triangle $A'B'C$. By the converse of Feuerbach's theorem, the circle (ω) is the incircle of triangle $A'B'C$, since it is tangent to sides $A'C$, $B'C$ and to the nine-point circle internally. Consequently $\rho = 2(s-c) = a+b-c$.

II. *Solution by M. A. Kirchberg, Hopkins, Mich.* The centers of the inscribed circle and the given circle being at (ρ, ρ) and $(a/2, b/2)$, respectively, on a coordinate system defined by the legs of the given triangle, where ρ is the unknown radius, we may simply equate the distance between centers to the difference in radii and solve, finding $\rho = a+b-c$.

Also solved by J. W. Baldwin, G. B. Charlesworth, P. A. Clement, Hüseyin Demir, I. A. Dodes, A. L. Epstein, R. L. Helmbold, A. R. Hyde, M. S. Klamkin, Viktors Linis, D. C. B. Marsh, Beckham Martin, R. K. Morley, T. F. Mulcrone, S. Parameswaran, C. F. Pinzka, P. W. A. Raine, G. B. Robison, David Rothman, Sister M. Stephanie, J. A. Tierney, Roy Westwick, and Roscoe Woods.

Morley studied the locus of ω when A and B are fixed and C varies.

Semi-vertical Angle of a Right Circular Cone

E 1142 [1954, 711]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.*

Find the semi-vertical angle of a right circular cone if three generating lines make angles of 2α, 2β, 2γ, with each other.

Solution by Leon Bankoff, Los Angeles, Calif. The sides of the base of the triangular pyramid determined by the three generating lines are $a = 2y \sin \alpha$, $b = 2y \sin \beta$, $c = 2y \sin \gamma$, where y is the slant height of the cone. The radius of the base of the cone is given by

$$R = abc/4\sqrt{s(s-a)(s-b)(s-c)},$$

where s is the semi-perimeter of the base of the pyramid. Since the semi-vertical angle ϕ is equal to arc sin R/y, we obtain

$$\phi = \arcsin \frac{2 \sin \alpha \sin \beta \sin \gamma}{\sqrt{(\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma)^2 - 2(\sin^4 \alpha + \sin^4 \beta + \sin^4 \gamma)}}.$$

Also solved by A. Buckley, G. B. Charlesworth, H. K. Crowder, Hüseyin

Demir, A. L. Epstein, R. L. Helmbold, A. S. Hendler, Raymond Huck, A. R. Hyde, Viktors Linis, D. C. B. Marsh, R. K. Morley, C. S. Ogilvy, S. Parameswaran, Walter Penney, Azriel Rosenfeld, D. C. Russell, Chih-yi Wang, Roy Westwick, R. H. Wilson, Jr., Roscoe Woods, and the proposer.

Demir gave the equivalent answer

$$\sin^2\phi = -\frac{16\sin^2\alpha\sin^2\beta\sin^2\gamma}{\begin{vmatrix} 0 & \sin\alpha & \sin\beta & \sin\gamma \\ \sin\alpha & 0 & \sin\gamma & \sin\beta \\ \sin\beta & \sin\gamma & 0 & \sin\alpha \\ \sin\gamma & \sin\beta & \sin\alpha & 0 \end{vmatrix}}.$$

Helmbold considered the analogous problem in an n-dimensional Euclidean vector space.

Concerning the Coefficients of a Polynomial

E 1143 [1954, 711]. *Proposed by C. D. Olds, San Jose State College, Calif.*

If the roots of the equation

$$a_0x^n - na_1x^{n-1} + \tfrac{1}{2}n(n-1)a_2x^{n-2} - \cdots + (-1)^n a_n = 0$$

are positive and distinct, prove that $a_ra_{n-r} > a_0a_n$, $r = 1, 2, \cdots, n-1$.

Solution by E. P. Starke, Rutgers University. With no additional difficulty we can prove more, namely, $a_pa_q > a_ra_s$ for all p, q, r, s with $p+q=r+s$, $|p-q| < |r-s|$. Since the given equation has n distinct positive roots, so also has the reciprocal equation

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}a_ix^i = 0.$$

Furthermore, by the law of the mean, each of the equations obtained by differentiation has $n-1$ distinct positive roots,

$$\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}a_ix^{n-i-1} = 0, \qquad \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}a_{n-i}x^{n-i-1} = 0.$$

Since the proposed conclusion is obviously true for $n=2$, we proceed by induction, assuming $a_pa_q > a_ra_s$ is true whenever $p+q=r+s$, $|p-q| < |r-s| \leqq n-1$. Hence

$$a_ka_{n-k} > \cdots > a_2a_{n-2} > a_1a_{n-1},$$

where $k = [n/2]$. Further, $a_1^2 > a_0a_2$ and $a_2a_{n-1} > a_1a_n$ imply, since the a's are positive, that

$$a_1a_{n-1} > a_0a_n,$$

which completes the proof.

The example $16x^3-64x^2+81x-34=0$ shows that the converse does not hold.

Also solved by I. A. Dodes, H. M. Feldman, N. J. Fine, A. J. Goldman, M. S. Klamkin, Viktors Linis, S. Parameswaran, Azriel Rosenfeld, Georgia Smith, and O. E. Stanaitis. Late solutions by A. E. Danese, C. F. Pinzka, and Albert Wolinsky.

A Logarithmic Inequality

E 1144 [1954, 711]. *Proposed by A. S. Hendler, Rensselaer Polytechnic Institute, N. Y.*

For what positive values of a is $\log_a b<b$ for all positive b?

Solution by F. R. Olson, University of Buffalo. If $a<1$, then $b=a$ would imply $1<a$. It follows that $a>1$. Consequently the inequality is equivalent to $b<a^b$ or $b^{1/b}<a$. By the usual derivative process the maximum of $b^{1/b}$ is found to occur when $b=e$. Hence $\log_a b<b$ for $a>e^{1/e}$.

Also solved by R. H. Ayres, Jr., P. T. Bateman, B. H. Bissinger, Julian Braun, G. B. Charlesworth, H. K. Crowder, Hüseyin Demir, I. A. Dodes, N. J. Fine, D. S. Greenstein, R. L. Helmbold, A. R. Hyde, M. S. Klamkin, Viktors Linis, D. C. B. Marsh, Morris Morduchow, T. F. Mulcrone, C. S. Ogilvy, M. W. Oliphant, S. Parameswaran, Walter Penney, L. L. Pennisi and N. C. Scholomiti (jointly), Azriel Rosenfeld and L. I. Sherry (jointly), D. C. Russell, C. M. Sandwick, Sr., O. E. Stanaitis, D. R. Sudborough, T. H. Sumner, Roy Westwick, R. H. Wilson, Jr., and the proposer. Late solution by C. F. Pinzka.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4639 [1955, 371]. *Correction.* For $k\geqq j$, read $k\neq j$.

4643. *Proposed by Norman Anning, Alhambra, California*

If r is a prime of the form $4k+1$ or is the product of such primes, show that the number of points with integral coordinates on the sphere

$$x^2+y^2+z^2=r^2$$

is $6r$.

4644. *Proposed by Azriel Rosenfeld, Columbia University*

Prove that in no field K can the additive group K^+ be isomorphic to the multiplicative group K^*.

4645. *Proposed by Leonard Carlitz, Duke University*

Let $0 \leqq r \leqq p-1$, where p is prime, and let $a_0, a_1, \cdots, a_r$ be integers not divisible by p. Define the determinant of order $r+1$

$$\Delta_r = \left| a_i^{p^j} \right| \qquad (i, j = 0, 1, \cdots, r).$$

Show that

$$\Delta_r \equiv 0 \pmod{p^{r(r+1)(r+2)/6}}.$$

4646. *Proposed by Oliver Gross, the Rand Corporation*

Let $F_1(x) = G_1(x) = x$ for x real, and for $n > 1$ define F_n, G_n recursively by

$$F_n(x_1, \cdots, x_n) = \max\,(x_1, G_{n-1}(x_2, \cdots, x_n)),$$
$$G_n(x_1, \cdots, x_n) = \min\,(x_1, F_{n-1}(x_2, \cdots, x_n)).$$

Are the following relations true:

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 F_n(x_1, \cdots, x_n)\,dx_1 \cdots dx_n = \frac{\pi\sqrt{3}}{9},$$

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 G_n(x_1, \cdots, x_n)\,dx_1 \cdots dx_n = 1 - \frac{\pi\sqrt{3}}{9}\,?$$

4647. *Proposed by James Munkres, Los Alamos Scientific Laboratory*

A permutation of the integers $1, \cdots, n$ is called an n-chain; two n-chains are disjoint if any two integers which are adjacent in one chain are not adjacent in the other. (The first integer is considered adjacent to the last for this purpose.) Does there exist, for each n, a collection containing $[(n-1)/2]$ mutually disjoint n-chains?

SOLUTIONS

4553 [1955, 259–260]. *Correction.*

Change the number in the last line from 1.12 to $6^{1/2} - 2^{1/2} = 1.035$.

Asymptotic Summation Formulas

4582 [1954, 199]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Show that

(a) $$\sum_{n=1}^{p} \frac{S_{1,n}^2}{n} \sim \frac{5}{3} \sum_{n=1}^{\infty} \frac{1}{n^3} + \frac{1}{3}(\gamma + \log p)^3,$$

(b) $$\sum_{n=1}^{p} \frac{S_{2,n}}{n} \sim \frac{\pi^2}{6}(\gamma + \log p) - \sum_{n=1}^{\infty} \frac{1}{n^3},$$

where γ is Euler's constant and $S_{m,n} = \sum_{r=1}^{n} r^{-m}$.

Solution by Leonard Carlitz, Duke University. We shall show that

(1) $$A = \sum_{n=1}^{k} \frac{1}{n} S_{2,n} = \frac{\pi^2}{6}(\gamma + \log k) - \sum_{n=1}^{\infty} \frac{1}{n^3} + O\left(\frac{\log k}{k}\right),$$

(2) $$B = \sum_{n=1}^{k} \frac{1}{n} S_{1,n}^2 = \frac{5}{3} \sum_{n=1}^{\infty} \frac{1}{n^3} + \frac{1}{3}(\gamma + \log k)^3 + O\left(\frac{\log k}{k}\right),$$

where $S_{m,k} = \sum_{n=1}^{k} n^{-m}$.

We recall that

(3) $$S_{1,k} = \log k + \gamma + O\left(\frac{1}{k}\right), \qquad S_{2,k} = \frac{\pi^2}{6} + O\left(\frac{1}{k}\right).$$

We have

(4) $$S_{1,k} S_{2,k} = A + C,$$

where

$$C = \sum_{n=1}^{k} \frac{1}{n} \sum_{r=n+1}^{k} \frac{1}{r^2} = \sum_{r=2}^{k} \frac{1}{r^2} \sum_{n=1}^{r-1} \frac{1}{n} = \sum_{r=2}^{k} \frac{1}{r^2} \sum_{n=1}^{r-1} \frac{1}{r-n}.$$

Thus

$$2C = \sum_{r=2}^{k} \frac{1}{r^2} \sum_{n=1}^{r-1} \left(\frac{1}{n} + \frac{1}{r-n}\right) = \sum_{r=2}^{k} \frac{1}{r} \sum_{n=1}^{r-1} \frac{1}{n(r-n)},$$

$$2C + \sum_{r=2}^{k} \frac{1}{r} \sum_{n=1}^{r-1} \frac{1}{n^2} = \sum_{r=2}^{k} \sum_{n=1}^{r-1} \frac{1}{n^2(r-n)} = \sum_{n=1}^{k-1} \frac{1}{n^2} \sum_{r=n+1}^{k} \frac{1}{r-n} = \sum_{n=1}^{k-1} \frac{1}{n^2} \sum_{r=1}^{k-n} \frac{1}{r}$$

$$= \sum_{r=1}^{k-1} \frac{1}{r} \sum_{n=1}^{k-r} \frac{1}{n^2} = \sum_{r=1}^{k-1} \frac{1}{r} \left\{\frac{\pi^2}{6} + O\left(\frac{1}{k-r}\right)\right\}$$

$$= \frac{\pi^2}{6} S_{1,k-1} + O\left(\sum_{r=1}^{k-1} \frac{1}{r(k-r)}\right).$$

But

$$\sum_{r=1}^{k-1} \frac{1}{r(k-r)} = \frac{1}{k} \sum_{r=1}^{k-1} \left(\frac{1}{r} + \frac{1}{k-r}\right) = \frac{2}{k} S_{1,k-1} = O\left(\frac{\log k}{k}\right),$$

so that

$$2C + A = \frac{\pi^2}{6} S_{1,k} + S_{3,k} + O\left(\frac{\log k}{k}\right).$$

Thus (4) becomes

$$2S_{1,k}S_{2,k} = 2A + 2C = A + \frac{\pi^2}{6}S_{1,k} + S_{3,k} + O\left(\frac{\log k}{k}\right).$$

Using (3) and

$$S_{3,k} = \sum_{n=1}^{\infty} \frac{1}{n^3} + O\left(\frac{1}{k^2}\right),$$

we get

$$2\left\{\log k + \gamma + O\left(\frac{1}{k}\right)\right\}\left\{\frac{\pi^2}{6} + O\left(\frac{1}{k}\right)\right\}$$
$$= A + \frac{\pi^2}{6}\left\{\log k + \gamma + O\left(\frac{1}{k}\right)\right\} + \sum_{n=1}^{\infty} \frac{1}{n^3} + O\left(\frac{\log k}{k}\right),$$

which reduces to (1).

In the next place

$$S_{1,k}^2 = B + DS_{1,k} + E, \tag{5}$$

where

$$D = \sum_{n=1}^{k} \frac{1}{n} \sum_{s=n+1}^{k} \frac{1}{s}, \qquad E = \sum_{n=1}^{k} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{r} \sum_{s=n+1}^{k} \frac{1}{s}.$$

Now

$$S_{1,k}^2 = 2D + S_{2,k}. \tag{6}$$

As for E, we have

$$E = \sum_{s=2}^{k} \frac{1}{s} \sum_{n=1}^{s-1} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{r}$$
$$= \sum_{s=1}^{k} \frac{1}{s} \sum_{n=1}^{s} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{r} - \sum_{s=1}^{k} \frac{1}{s^2} \sum_{r=1}^{s} \frac{1}{r}$$
$$= \sum_{s=1}^{k} \frac{1}{s} \sum_{r=1}^{s} \frac{1}{r} \sum_{n=r}^{s} \frac{1}{n} - \sum_{s=1}^{k} \frac{1}{s^2} \sum_{r=1}^{s} \frac{1}{r},$$

so that

$$2E = \sum_{s=1}^{k} \frac{1}{s}\left(\sum_{r=1}^{s} \frac{1}{r}\right)^2 + \sum_{s=1}^{k} \frac{1}{s} \sum_{r=1}^{s} \frac{1}{r^2} - 2\sum_{r=1}^{k} \frac{1}{r} \sum_{s=r}^{k} \frac{1}{s^2} \tag{7}$$
$$= B - 2S_{1,k}S_{2,k} + 3A - 2S_{3,k}.$$

Thus using (5), (6), (7) we get

$$S_{1,k}^3 = \frac{3}{2}B + \frac{1}{2}(S_{1,k}^2 - S_{2,k})S_{1,k} - S_{1,k}S_{2,k} + \frac{3}{2}A - S_{3,k},$$

which reduces to

$$\begin{aligned} S_{1,k}^3 &= 3B - 3S_{2,k}S_{1,k} + 3A - 2S_{3,k} \\ &= 3B - 3(S_{2,k}S_{1,k} - A - S_{3,k}) + 5S_{3,k}. \end{aligned}$$

Now using (1) and (3) we immediately get (2).

Also solved by the Proposer.

A Set of Transcendental Numbers

4583 [1954, 263]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

For $a_n = 1$, $n = 1, 2, \cdots$,

$$S = \sum_{r=1}^{\infty} [r!]^{-a_r}$$

is transcendental. Find other non-decreasing sequences $\{a_n\}$ such that S is transcendental.

Solution by the Proposer. Let

$$\sum_{r=1}^{n} [r!]^{-a_r}$$

be an approximation to S. If S is algebraic, then by the theorem of Liouville on algebraic numbers

$$\sum_{r=n+1}^{\infty} [r!]^{-a_r} \geqq [n!]^{-ma_n} \tag{1}$$

must hold for $m > 2$ and n sufficiently large. If we take $a_{j+1} > 1 + a_j$, (1) implies

$$\frac{[(n+1)!]^{-a_{n+1}}}{1 - \dfrac{1}{(n+1)!}} \geqq [n!]^{-ma_n}$$

which, in turn, implies that

$$2[n!]^{ma_n} \geqq [(n+1)!]^{a_{n+1}}. \tag{2}$$

But (2) will not hold if we choose a_n of sufficiently high order in n. In fact, since $2[n!]^m < [(n+1)!]^{n+1}$ for $n > m$, we have

$$2^{n!}[n!]^{n!} < [(n+1)!]^{(n+1)!}$$

which contradicts (2) with $a_n = n!$ and hence also contradicts (1). Thus S is transcendental for $a_n = n!$ and also for any a_n of equivalent or higher order, such as $(n!)^p$, n^n, *etc.*

Approximations for π and for e

4584 [1954, 263]. *Proposed by W. F. Cheney, Jr., University of Connecticut*

The equation $15x^2 - 78x + 97 = 0$ has a root 3.14160+, and the equation $27x^2 - 55x - 50 = 0$ has a root 2.71829+. It is required to find quadratic equations with integer coefficients, each less than 100 in absolute value, which have roots closest to π and e, respectively.

Solution by Walter Penney, Washington, D. C. If π is a root of the equation $ax^2 + bx + c = 0$, we must have $a\pi^2 + b\pi + c = 0$. Since a can be taken positive without any loss of generality, we seek integers $a (0 < a < 100)$ and $b (-100 < b < 100)$ such that $9.8696044010 \cdots a + 3.1415926535 \cdots b$ is as close as possible to an integer $c (-100 < c < 100)$; a must be less than 42, since $42\pi^2 - 100\pi$ is greater than 100. If a is 41, b must be 99, 98 or 97. If a is 40, $94 \leqq b \leqq 99$, *etc.* When a value of c close to an integer is found, $7\pi (= 21.9911485745 \cdots)$ can be added to or subtracted from this value to see whether the result is improved. In this way we find the following pairs of values of (a, b) which give near-integral values of c: (36, −94), (28, −66), (27, −74), (26, −82), (19, −39), (15, −78), (11, −4). Testing each in turn, we find the best result is $19x^2 - 39x - 65 = 0$ with the root 3.14158805.

Similarly we find $35x^2 - 83x - 33 = 0$ has a root 2.71828582.

Also solved by J. P. Ballantine, Bart Park, and the Proposer.

A Series Involving the Partition Function

4585 [1954, 263]. *Proposed by H. F. Sandham, Institute for Advanced Studies, Dublin, Ireland*

Denoting the number of partitions of n by $p(n)$, prove

$$\sum_{n=1}^{\infty} \frac{np(n)}{\cosh \pi\sqrt{(2n - \frac{1}{4})}} = \frac{1}{4\pi}.$$

Solution by Chih-yi Wang, University of Minnesota. This solution was suggested by the Proposer's paper *Two series of partitions* (THIS MONTHLY, v. 61 (1954), pp. 104–106), where we have the relations

$$(1) \qquad \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots} = \sum_{n=0}^{\infty} p(n) q^n,$$

$$(2) \qquad \int_0^1 q^{\beta+1/8}\{(1-q)(1-q^2)(1-q^3)\cdots\}^3 \frac{dq}{q} = \frac{2\pi}{\cosh \pi\sqrt{2\beta}},$$

$(\beta > -1/8)$. By putting $\beta = n - 1/8$ in (2), we get

$$\int_0^1 \{(1-q)(1-q^2)(1-q^3)\cdots\}^3 d(q^n) = \frac{2\pi n}{\cosh \pi\sqrt{(2n-\frac{1}{4})}}.$$

Multiplying across by $p(n)$ and summing with respect to n from 1 to ∞, we have, by aid of (1),

$$2\pi \sum_{n=1}^{\infty} \frac{np(n)}{\cosh \pi\sqrt{(2n-\frac{1}{4})}}$$

$$= \int_0^1 \{(1-q)(1-q^2)(1-q^3)\cdots\}^3 d \frac{1}{(1-q)(1-q^2)\cdots}$$

$$= -\frac{1}{2}\left[\{(1-q)(1-q^2)\cdots\}^2\right]_0^1 = \frac{1}{2}.$$

Hence the stated result follows.

Also solved by the Proposer.

Behavior of Improper Integrals

4587 [1954, 264]. *Proposed by A. E. Livingston, University of Washington, and Harry Pollard, Cornell University*

Show that

$$-(2n+1)\pi \int_{(2n+1)\pi}^{\infty} x^{-1} \sin x dx \uparrow 1.$$

(A similar statement holds for

$$2n\pi \int_{2n\pi}^{\infty} x^{-1} \sin x dx.)$$

Solution by Chih-yi Wang, University of Minnesota. Let the given integral be denoted by $I(n)$. Using the formula for integration by parts, we obtain

$$I(n) = 1 + (2n+1)\pi \int_{(2n+1)\pi}^{\infty} x^{-2} \cos x dx.$$

Now consider $I(n)$ as a continuous function of the positive real variable n. We get, by differentiating with respect to n,

$$I'(n) = \frac{2}{2n+1} + 2\pi \int_{(2n+1)\pi}^{\infty} x^{-2} \cos x dx$$

$$\geqq \frac{2}{2n+1} - 2\pi \int_{(2n+1)\pi}^{\infty} x^{-2} dx = 0,$$

which implies that $I(n) \uparrow$. The proof will be complete if we integrate by parts three times, then

$$I(n) = 1 - 2(2n + 1)^{-2}\pi^{-2} + O(n^{-3}),$$

which approaches 1 as $n \to \infty$.

The second proposed statement is established by the same procedure.

Also solved by D. L. Greenstein, M. S. Klamkin, M. R. Spiegel, and the Proposers.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio

Theory of Games and Statistical Decisions. By David Blackwell and M. A. Girshick. New York, John Wiley and Sons, Inc., 1954. ix+355 pages. $7.50.

As the first textbook to appear on Wald's theory of statistical decision functions, Blackwell and Girshick's book ought to attract the interest of the general mathematician without a specific interest in research in mathematical statistics. It is Wald's achievement to have placed the problems of mathematical statistics within the frame of a precise axiomatic mathematical theory and through this transformation to have made significant practical advances in the field through the development of sequential analysis and sequential decision theory. This is as straightforward an illustration of the power of the abstract mathematical method as anyone could wish.

The two basic principles underlying Wald's work are first, that a statistical procedure should be tested by its consequences under various circumstances as measured by the expected risk function, and second, that for a procedure with a variable number of steps, part of the risk consists in the cost of the steps. From the first of these principles, it follows that statistical decision theory can be set in the framework of the Von Neumann theory of the normal form of a zero-sum two-person game. It follows from the absence of a specific conscious opponent in the statistical "game" that the objects of interest and consequently the mathematical problems may be different in the two theories, but both fall under the aegis of the theory of convex sets and functions in linear spaces. Chapters 1 and 2 give a very clear and complete presentation of the theory of two-person zero-sum games. The formal description of statistical and sequential games is presented in Chapter 3, while Chapter 4 on principles of choice indicates why the study of the minimax strategy solutions of these games is inadequate from the point of view of the statistical theory. The formal theory of more

general classes of optimal strategies of significance from this viewpoint, particularly the admissible and Bayes strategies, is developed in Chapter 5. The remaining seven chapters apply these general concepts to the determination of the Bayes strategies for statistical games under various special assumptions, and in particular for sequential games in Chapters 9 and 10.

The amount of mathematical knowledge explicitly presupposed by the book is fairly limited as far as elementary analysis and linear algebra are concerned. What is presupposed, however, is not so much knowledge as familiarity with the whole process of abstract reasoning in the framework of elementary set theory. The great clarity of the presentation is achieved by taking full advantage of the notation and conventions of abstract mathematics as commonly practiced at the present time. If the first two chapters on game theory are compared to the related sections of McKinsey's book, one is struck both by the absence of discursive material in the present book and by its enormously greater mathematical sophistication. It is curious that one could read this whole book on statistics without knowing any but the most elementary facts about probability theory and without once meeting Stirling's formula. On a different tack, it might be mentioned that the reviewer was puzzled by the fact that, in the midst of extremely long formal definitions and proofs, the authors chose to use the property of compactness of sets in Euclidian spaces without ever defining the term and in the process made a number of petty but inelegant mis-statements. The book is singularly free of printer's errors and the only mathematical mistake the reviewer could find was on page 154, where a qualitative discussion correct for discrete distributions is incorrectly extrapolated to continuous distributions.

Perhaps the pressure of books as important as this one will aid in the modernization of the undergraduate mathematical curriculum in American colleges to the extent that it may be possible in the future to use this book on a wide scale, as the authors' preface states they intend it, as a textbook for first-year graduate students.

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Differential and Integral Calculus. By C. E. Love and E. D. Rainville. Fifth Edition. The Macmillan Company, New York, 1954. vi+526 pages. $5.75.

Calculus. By G. E. F. Sherwood and A. E. Taylor. Third Edition. New York, Prentice-Hall, Inc., 1954. vii+579 pages. $7.65.

Calculus, an Introduction to Analysis, and A Tool for the Sciences. By G. M. Merriman. Henry Holt and Company, New York, 1954. viii+625 pages. $6.50.

These three texts are alike in many ways: each is as long as an historical novel, as rich as a well-made fruitcake, and as carefully organized as a mountain-climbing expedition. Starting with some work on limits, each of the books develops differentiation of algebraic functions, mentioning the process of anti-differentiation fairly early. Other topics considered (not in this order in all the

books) are differentials, the mean-value theorem, the definite integral, the elementary transcendental functions, technique of integration, partial differentiation, applications to problems of geometry and physics in two and three dimensions, series, approximate integration, and differential equations. And lots mo'. There is an abundance of excellent problems in each book; some of the problems are, of course, the old stand-bys ("A Norman window consists . . . ") but many have interesting novel features.

The texts differ, one from another, with respect to rigor of treatment, formality of tone, degree of explicit motivation in the introduction of topics, and "reconditeness," to borrow a word from Whitehead's essay, *The Mathematical Curriculum*. One way to convey a feeling for the flavor of the books is to present some quotations from each, with relatively little comment. The limitations of such a method are obvious; nevertheless this reviewer feels that the following quotations present a faithful picture of the spirit of the books.

I. Limits

1. *L. and R., p. 9.* "Let $f(x)$ be a function of x and let a be constant. If there is a number L such that, *in order to make the value of $f(x)$ as close to L as may be desired, it is sufficient to choose x close enough to a, but different from a*, then we say that the limit of $f(x)$, as x approaches a, is L." This statement is followed by a page and a half of elaboration. P. 495 (appendix), "Definition of a limit. We say that $\lim_{x\to a} f(x) = L$, if for every positive number ϵ (arbitrarily small), there exists a number δ such that, in order to make $|f(x)-L| < \epsilon$, it is sufficient that x satisfy $|x-a| < \delta$, $x \neq a$."

2. *S. and T.* After two pages of discussion of derivatives, involving an intuitive notion of a limit, we find, on p. 22, "As yet we have not given a formal or precise definition of what this means . . . We are considering some particular function, and we ask the question: Does $f(x)$ approach some limit as x approaches x_0? To answer this, we inspect the table, looking at all the values of x near x_0. *Perhaps the inspection will show us that there is a certain number A with the property that $f(x)$ differs from A by very little when the difference between x and x_0 is small, and that the difference $f(x)-A$ can be brought down to any desired smallness, and maintained that small or even smaller, simply by requiring x to be near enough to x_0. If so, this is exactly what we mean by saying that $f(x)\to A$ as $x\to x_0$.*" This statement is followed by a page and a half of elaboration, which includes the following "brief formal statement of the definition of a limit": "Definition. We write $\lim_{x\to x_0} f(x) = A$ and say that $f(x)\to A$ as $x\to x_0$, if corresponding to each positive number ϵ there is some positive number δ such that $|f(x)-A| < \epsilon$ whenever $0 < |x-x_0| < \delta$."

3. *M.* After four pages of discussion of the instantaneous velocity of a vertically thrown ball, we have, p. 15, the following: "Definition: If a variable v moves on its range R so that all values assumed by v after a certain one lie arbitrarily close to a constant l (not necessarily on R), then l is the *limit* of v: $l = \lim v$." After two pages of elaboration, and one page dealing with functions,

we come to the following: "Definition. If (and only if) for arbitrary measure $\epsilon>0$ a corresponding dependent measure $\delta>0$ can be exhibited such that for all x on the interval $0<|x-a|<\delta$ the functional values $f(x)$ satisfy $0<|f(x)-L|<\epsilon$, then the number L is the *limit* (*value*) of $f(x)$ as $x\to a$."

It seems clear that S. and T. win this round.

II. Plane Areas

1. *L. and R., pp. 112–120.* "Calculus grew out of the attempts . . . of mathematicians to solve two major problems . . . The second problem was to obtain the area bounded by a curve $y=f(x)$, the x-axis, and two ordinates $x=a$ and $x=b$.

"This second problem is solved by a judicious extension of the elementary concept of the area of a rectangle as the product of its base and its altitude.

"In Fig. 57, let the interval $a\leqq x\leqq b$ be divided into n parts in any manner . . .

"It is reasonable that, if the maximum width of the rectangles shown be taken sufficiently small, and the number of rectangles correspondingly large, then the sum of the areas of the rectangles will approximate, as closely as desired, a quantity which agrees with our intuitive concept . . . of the required area.

"Therefore, we proceed to lay down as our definition of the area A . . . the following:

$$A=\lim_{\max \Delta x_k\to 0}\sum_{k=1}^{n} f(x_k)\Delta x_k,$$

in which, since the widths Δx_k of the rectangles approach zero, the number of them, n, must approach infinity.

"At once we are confronted with the question of whether the limit . . . exists, and, if it does exist, with the problem of determining how to compute that limit . . . "

The foregoing exposition is followed by a statement of the Fundamental Theorem of the Integral Calculus, and then:

"Proof of the Fundamental Theorem will be omitted . . . Here we content ourselves with the discussion in the next section, a treatment intended to make the Fundamental Theorem plausible, not to prove it."

Next come five pages concerned with an intuitive discussion of the Fundamental Theorem, and with the computation of plane areas.

With the authors' premises about the appropriate degree of depth and rigor, it is hard to imagine a better exposition than the foregoing.

2. *S. and T., pp. 28–31.* "The problem of calculating areas bounded all or in part by curved lines is a fundamental problem of geometry. Let us consider the problem for the area OAB shown in Fig. 7 where the bounding curve is $y=x^2$. This problem was solved by Archimedes in the third century B.C.

"Area is measured by comparing the given figure with a standard figure, the unit area, which is a square with sides one unit long. The formula for the area

of a rectangle is deduced from such a comparison. In dealing with curved figures, we may attempt to choose a set of rectangles which do not overlap and which nearly cover the figure. An approximation to the area of the figure is then furnished by the total area of the rectangles. This is the basis for our procedure."

After a page devoted to algebraic development, we find:

"Hence we can see that S_n approaches $\frac{1}{3}c^3$ as a limiting value. We conclude that the area between the curve $y=x^2$ and the x-axis, from $x=0$ to $x=c$, is $S=\frac{1}{3}c^3$." (Note that area has not yet been defined.)

On the following page, the problem of area is considered for an arbitrary function which is continuous and never negative on the interval $a \leqq x \leqq b$: " . . . We then consider the sequence $S_1, S_2, S_3, \cdots, S_n, \cdots$. It always turns out that, as n is made larger and larger, the numbers S_n approach a limiting value S. This number S is, *by definition*, taken to be the area of the figure . . . "

Pp. 63–66. "Finding areas by antidifferentiation. . . . "

Pp. 246–258. These pages are devoted to a discussion of lower and upper sums, to re-definitions of the area under a curve and of the area between two curves, and to proofs of the mean-value theorem for integrals and of the Fundamental Theorem.

3. *M., pp. 206–232.* Starting with the problem of finding the work done in emptying a tank, the author is led to the geometrical problem of finding "a numerical value for an area. But the 'area problem' is a new problem. . . . To extend the notion of area we must lead off from known ground. In terms of the area unit of the plane content of a 1×1 square, the reader knows . . . This is the extent of his basic information. It is not fair to include here the familiar area of a circle . . . Any extension, then, of the concept of area and calculation of area must of mathematical necessity be based on the fundamental area of rectangle. In a larger sense it can be pointed out that we actually need to *define* what is meant by the area enclosed by one or more *curved* arcs. Such assignment of meaning in terms of *square* units presents difficulties. The present goal is the invention of a suitable extended definition of area; the result will also furnish a calculus of its numerical value."

Then follow three pages of discussion of "Area and Perimeter of a Circle in Greek Geometry." Next comes the statement: "As in so many extensions of old notions toward new fields of conquest, we first review an old problem in which we know the answer, but seek a generalizable method of obtaining it." Four pages are devoted to the calculation of sums of areas of rectangles inscribed in $y=2x$ and in $y=12x-x^2$, and of the limits of these sums. "Yet, from the intuitive geometry of the situation, any member of $\{S_n\}$ *encloses* the area OAB sought, while any member of $\{s_n\}$ *is enclosed by* this area. Thus OAB is caught between $\{S_n\}$ and $\{s_n\}$, which sequences *both* approach 144; how, then, can we mistrust the appropriation of this common limit $S=144$ as the definition of area and area-number?"

The definite integral over a closed range of a single-valued function is next defined, the existence of the definite integral for continuous functions is stated,

the area bounded by the graph of a continuous non-negative function over $a \leqq x \leqq b$ and the lines $y=0$, $x=a$, $x=b$, is defined, and the Fundamental Theorem is stated and proved, using the mean-value theorem and the existence of the definite integral for continuous functions.

III. The Area of a Surface of Revolution

1. *L. and R., pp. 305–306.* " . . . By elementary geometry, the surface area of this conical frustum is *the circumference of the middle section multiplied by the slant height*, or $2\pi(y_i+\frac{1}{2}\Delta y_i)\Delta s'_i$. Discarding higher order infinitesimals as usual, . . . "

2. *S. and T., pp. 321–327.* " . . . We can write this sum as two separate sums . . . Each of these is a sum to which the formula of Bliss . . . applies . . . " Proof of the formula of Bliss is omitted.

3. *M., pp. 274–279.* " . . . Bliss' Theorem is easy to understand and to apply; it is modernly finding its way into the literature of teaching elementary calculus. We shall accept the theorem without proof, but we attempt to make it plausible at the end of this section. . . . "

Each of these books has so many good features that the reviewer is tempted to suggest a liaison of *L*, *R*, *S*, *T*, and *M* to produce one text incorporating the advantages of all. Mindful, however, of Bernard Shaw's retort to Isadora Duncan, he resists the temptation and accepts these books for what they are: excellent additions to the library of American texts.

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Trigonometry. By Roy Dubisch. New York, The Ronald Press, 1955. xiv+283 +Index+tables. $5.00.

Trigonometry. By E. P. Vance. Cambridge, Massachusetts, Addison-Wesley Publishing Company, 1954. viii+140+Index+tables. $3.00.

For years the teacher of mathematics has been inundated by elementary textbooks which, in mathematical content and presentation, might well have been printed from the same century-old plates. It is a pleasure to report that neither of the books under review falls into this category. Both evidence sincere attempts to modernize the teaching of trigonometry and to present the subject as part of the main stream of mathematics.

The Vance book is a compact (124 pages of text) volume which the instructor would have to amplify in class. Professor Vance presents trigonometry just as a mathematician thinks of the subject, his emphasis clearly upon the mathematical aspects. The unity and clarity thus achieved are remarkable. There is some question, however, in regard to its use as a text. Will the student be as enthusiastic over the Vance book as is his teacher?

After a first chapter on Numbers and Coordinate Systems, Vance proceeds to circular functions. His very insistence upon this term seems to leave open the

possibility of other functions related to conic sections. The third chapter, Functions Involving More Than One Angle, contains the first completely satisfactory derivation of composite angle formulas to be found in an elementary text. Chapter 4, Solution of Triangles, is very neatly done, properly beginning with the law of sines and only then specializing to right triangles. Chapter 5, Inverse Functions and Graphs, and Chapter 6, Identities and Equations, are both good but not unusual. Chapter 7, Complex Numbers, begins with a discussion of the algebra of number pairs and continues to exploit the quite general approach. The last chapter, Applications of Circular Functions to Periodic Phenomena, is wonderful, even going so far as to talk about harmonic analysis. It should whet the appetite of even a mildly interested student. Finally, it should be noted that Vance does not include the study of logarithms in his book except in an appendix.

In contrast to the brevity of the Vance book, which would place interesting demands upon the instructor, Professor Dubisch has written such an extensive and lucid treatment that an instructor would be almost superfluous. His book might well be labelled as "self-teaching." Still the presentation has much the same spirit as does the Vance book.

The introductory chapter contains a nice explanation of "function," reminiscent of that in Begle's recent calculus text. The second chapter deals with the arc-length function $ALF(ALF(x)$ is the point on the unit circle with coordinates $(\cos x, \sin x)$). This appears to be an artificiality but it leads to a very neat formulation. In Chapters 3 through 7, Dubisch takes up trigonometric functions, tables and graphs, inverse functions, identities and equations, all in terms of ALF. Then in Chapter 8, the functions are re-defined in terms of angles. This approach introduces radian measurement very naturally. Chapter 9 repeats the study of identities and equations, now in terms of angles. Solution of Triangles is the title of Chapter 10 and the material here is quite standard, as is the eleventh chapter on complex numbers. The final two chapters take up logarithms and are well done but unspectacular. (Why does the purely algebraic study of logarithms traditionally appear in a trigonometry course?) Briefly then, Dubisch starts out with a new and mathematically interesting approach. He follows his new line very well until he gets to the traditional material of trigonometry whereupon the treatment becomes quite standard. This is not to be construed entirely as a derogatory comment since a sound foundation is actually the bulk of the work in any construction.

The format of the Dubisch book is good but cannot quite compare with the excellent standards associated with Addison-Wesley.

Both authors and both publishers deserve praise for leaving the old and beaten track and heading toward the modern. It is to be hoped that others will be stimulated by these two books to write much-needed modern texts for other elementary courses.

J. G. HOCKING
Michigan State College

Mathematics in Western Culture. By Morris Kline. New York, Oxford University Press, 1953. 12+484 pages. $7.50.

The object of this book is to show that mathematics has been an important influence in the development of our Western culture. This might be relatively easy if the author were writing for mathematicians; but the foreword by Professor Courant implies that the book is addressed to the group of intelligent people who do not have a background of mathematical knowledge, and this makes the undertaking a formidable one. In his introductory chapter the author says, "The subject is not a series of techniques. These are indeed the least important aspect, and they fall as far short of representing mathematics as color mixing does of painting. The techniques are mathematics stripped of motivation, reasoning, beauty, and significance." He then proceeds to present mathematics stripped of techniques. The non-mathematical reader, without encountering any technique beyond a little elementary algebra and some simple graphs, may possibly get some notion of the cultural significance of the conception and growth of various mathematical ideas.

After the introduction one finds three chapters on Egyptian, Babylonian, and Greek mathematics, and it is pointed out that Euclid and the Greek philosophers represent the beginnings of abstract thinking and deductive reasoning. We are told that "the contribution of the Greeks that did most to determine the character of present-day civilization was their mathematics." Certainly Euclid's contribution to logical thinking could hardly be overestimated; but in emphasizing this point it was not necessary for the author to imply, somewhat uncritically, that Euclid deduced all his geometric theorems by flawless reasoning from just ten well chosen axioms.

Two chapters are devoted to the Alexandrian era, in which are found the beginnings of trigonometry, with practical applications to navigation, surveying, and such astronomical measurements as the radius of the earth and the distance to the moon. Then, under the heading "Interlude," a single chapter disposes of the nine or ten centuries of the medieval period in which there was no progress in mathematics and little or none in the development of Western culture.

The next six chapters describe the rapid mathematical advances of the two centuries beginning with the Renaissance. It is pointed out that the work of Copernicus, Kepler, and Galileo in astronomy did much to free scientific thought from the tyranny of the church; that the mathematical theory of perspective of Leonardo da Vinci and Dürer brought about important changes in the art of painting; and that Descartes' mathematical approach to the search for truth had a profound influence on religious and philosophical thinking.

There is a chapter on Newton's discovery of the law of gravitation, showing that it wiped out any distinction between celestial and terrestrial mechanics; and the author then devotes four chapters to the work of Leibniz and Newton on the calculus, and a description of the Newtonian influence on science, religion, literature, and aesthetics. In discussing the work of Leibniz and Newton the

author is distinctly more critical than he was in the case of Euclid's *Elements*. He says:

"No one can read the details of their writings on the calculus without being amazed by the variety of ways in which they stabbed at, around, and about the correct version of the limit concept without actually striking it. Several times they changed their approaches and contradicted their earlier statements. Neither man succeeded in doing more with the limit concept proper than confusing himself, his contemporaries, and even his successors."

Two chapters discuss the mathematical treatment of sound waves, of electromagnetic and light waves, and the effect of this work on music and the transmission of music and other sound.

The mathematical approach to the social sciences, especially through the application of the theories of statistics and probability, is discussed in three chapters. The author uses the device of a rather amusing Platonic dialogue between "Mr. Determinist" and "Mr. Probability of High Degree" to point out the conflict between the philosophy of determinism and the statistical point of view. The final chapters discuss the infinite and related paradoxes, non-Euclidean geometries and the belief in absolute truth, Einstein's relativity and the question of absolute time and space, and mathematics as itself a form of creative art.

Many mathematicians are skeptical with regard to books about mathematics written by mathematicians for the non-mathematical reader. The book under review makes a strong case in support of its thesis, and will be of very considerable interest *to mathematicians*. The reviewer would be interested, however, to read an appraisal of the book written by one of the non-mathematical laymen to whom it is supposedly addressed.

W. B. CARVER
Ithaca, New York

NEW BOOKS RECEIVED

American Men of Science, I, The Physical Sciences. New York, R. R. Bowker Company, Feb. 1955, $20.00.

An Introduction to the Theory of Numbers. By G. H. Hardy and E. M. Wright. Oxford, Clarendon Press, England, 1954. xvi+419 pages. 42s. Net.

Functional Mathematics, Grade 8. By W. A. Gager, D. H. Johnson, C. N. Shuster, R. Madden, F. W. Kokomoor. New York, Charles Scribner's Sons, 1955. viii+373 pages. $2.24.

Memorial des Sciences Mathematiques. Published by L'Academie des Sciences de Paris, Henri Villat, Director. Fascicule CXXIX, *Integration des equations aux derivees partielles du second ordre par la methode de Drach*. By M. Georges Heilbronn. Paris, Gauthier-Villars, 1955. 96 pages.

Automatic Feedback Control System Synthesis. By John G. Truxal. New York, McGraw-Hill Book Company, 1955. xiii+675 pages. $12.50.

Philosophy and Analysis. Edited by Margaret MacDonald. New York, Philosophical Library, 1954. 296 pages. $7.50.

Higher Transcendental Functions, Volume 3. Sponsored by California Institute of Technology, The Bateman Project Staff, Editor, A. Erdélyi. New York, McGraw-Hill Book Company, 1955.

One Hundred Mathematical Curiosities. By William R. Ransom. Portland, Maine, Box 1075, J. Weston Walch, Publisher, 1955. 212 pages.

Table of Salvo Kill Probabilities for Square Targets. By U. S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series 44. Washington, 25, D. C., 1955. ix+33 pages, 30 cents.

A Short Table for Bessel Functions of Integer Orders and Large Arguments. By L. Fox. Published for the Royal Society at the University Press, Cambridge, 1954. 27 pages. 6s.6d. net.

Royal Society Mathematical Tables, 3, Table of Binomial Coefficients. Edited by J. C. P. Miller. New York, Cambridge University Press, American Branch, 1954. 162 pages. $5.50.

Selected Papers in Statistics and Probability. By Abraham Wald. Edited by the Institute of Mathematical Statistics. New York, McGraw-Hill Book Company, 1955. 702 pages. $8.00.

Differential and Integral Calculus, 2nd Edition. By H. M. Bacon. New York, McGraw-Hill Book Company, 1955. 547 pages. $6.00.

Proceedings of the First Conference on Training Personnel for the Computing Machine Field, (Held at Wayne University, Detroit, Mich., June 22 and 23, 1954). Edited by Arvid W. Jacobson. Detroit, Wayne University Press, 1955. 104 pages. $5.00.

Partial Differential Equations. By I. G. Petrovsky. New York, Interscience Publishers, Inc., 1954. 245 pages. $5.75.

Existence Theorems for Ordinary Differential Equations. By F. J. Murray and K. S. Miller. New York, New York University Press (Distributed by Interscience Publishers, New York), 1954. 154 pages. $5.00.

The Elements of Probability Theory and Some of its Applications. By Harald Cramer. New York, John Wiley and Sons, Inc., 1955. 281 pages. $7.00.

An Introduction to Deductive Logic. By Hugues Leblanc. New York, John Wiley and Sons, Company, 1955. 244 pages. $4.75.

Advanced Mathematics for Engineers, 3rd Ed. Revised by F. H. Miller. New York, John Wiley and Sons, Inc., 1955. 548 pages. $6.50.

Modern Trigonometry. By J. C. Brixey and R. V. Andree. New York, Henry Holt and Company, 1955. 209 pages.

Science and the Human Imagination, Aspects of the History and Logic of Physical Science. By Mary B. Hesse. New York, Philosophical Library, 1955. 171 pages. $3.75.

Introduction to College Mathematics, Revised Edition. By M. A. Hill, Jr., and J. Burton Linker. New York, Henry Holt and Company, 1955. 428 pages +99 pages of tables. $5.25.

Plane Trigonometry. By C. R. Wylie, Jr., New York, McGraw-Hill Book Company, 1955. 381 pages. $4.00.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

TOPOLOGY NOTES BY A. W. TUCKER

Microfilms of Professor Tucker's notes on Elementary Combinatorial Topology, given at Haverford College under the Philips Fund, and also the notes of several lectures given at the University of Washington are available at *University Microfilms*, 313 N. First Street, Ann Arbor, Mich., at a total cost of $2.70. Requests should be sent directly to *University Microfilms*.

PERSONAL ITEMS

Professor F. E. Johnston of George Washington University was the official representative of the Association at the dedication of the Glenn L. Martin Institute of Technology at the University of Maryland on March 25, 1955.

Professor A. W. McGaughey of Bradley University represented the Association at the inauguration of President Harold P. Rodes of Bradley University on May 6, 1955.

Dr. A. Helen Tappan, who has retired from her position as Head of the Department of Mathematics of Western College for Women, was awarded the honorary degree of Doctor of Humane Letters by the College in June, 1954. Previously, she was cited by the Alumnae Association as one of its ten outstanding alumnae.

Franklin and Marshall College announces the following: Associate Professor D. W. Western has been promoted to a professorship; Assistant Professors V. H. Haag and J. R. Holzinger have been promoted to associate professorships.

Mr. D. L. Arenson, formerly a technical engineer for Cook Research Laboratory, Aerophysics Section, Skokie, Illinois, has a position as Manager of Ex-Cel Development Laboratory, Chicago, Illinois.

Associate Professor H. A. Bender of the University of Rhode Island has been promoted to a professorship.

Associate Professor A. B. Brown of Queens College has been promoted to a professorship.

Mr. J. S. Chesna, previously a physicist for Eastman Kodak Company, Rochester, New York, is President of the Infrared Electro-Optical Bolometer Company, Rochester, New York.

Associate Professor D. E. Coffey of Lawrence Institute of Technology has been appointed to an instructorship at Northwest Missouri State College.

Associate Professor Elsie T. Church of Northwestern State College has been promoted to a professorship.

Dr. W. J. Coles, formerly a Fulbright Scholar at King's College, Cambridge,

England, has been appointed to an instructorship at the University of Wisconsin.

Associate Professor P. R. Culwell, chairman of the Department of Mathematics, Trinity University, has been appointed to an instructorship at San Antonio College.

Mr. S. H. Dalrymple, formerly a student at the University of Texas, is now a research assistant at Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico.

Assistant Professor A. E. Danese of Western Reserve University has been appointed to an instructorship at the University of Tennessee.

Dr. T. C. Doyle, previously a mathematician at the Naval Research Laboratory, Washington, D. C., is now a member of the staff of Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

Dr. J. W. Forman of the University of Kansas has a position as an applied science representative for I.B.M. Corporation, Kansas City, Missouri.

Mr. E. P. Graney, recently a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California, has a digital computer programming and consulting service in Ann Arbor, Michigan.

Dr. T. R. Horton, applied science representative for I.B.M. Corporation, Atlanta, Georgia, has been transferred to Asheville, North Carolina.

Mr. W. R. Hydeman, formerly a mathematician with Engineering Research Associates, Arlington, Virginia, has accepted a position as sales engineer, Electronic Computer Department, Remington Rand, Inc., New York, New York.

Mr. A. R. Kneer, formerly a student at Gonzaga University, has a position as a mathematician in the Department of Commerce, Weather Bureau, Washington, D. C.

Mr. David Loev, previously an assistant development engineer at Burroughs Adding Machine Company, Philadelphia, Pennsylvania, is now an assistant scientist at the Weizmann Institute of Science, Rehovot, Israel.

Miss Anna E. Many, counselor to women at Newcomb Memorial College, Tulane University, has retired.

Mr. G. E. Meike, formerly a teaching fellow at the University of Detroit, has been promoted to an instructorship.

Mr. John Neufeld has been appointed to an instructorship at Detroit Institute of Technology.

Dr. C. V. Newsom, Associate Commissioner for Higher and Professional Education, State University of New York, has been named Executive Vice Chancellor of New York University, effective July 15, 1955.

Mr. R. D. Oeder, previously a mathematician at the Radiation Laboratory, University of California, Livermore Site, has accepted a position as an applied science representative, I.B.M. Corporation, Seattle, Washington.

Mr. J. H. Oppenheim, formerly a computer at the University of Chicago, is now a graduate student at the University of Illinois.

Mr. J. A. Painter, previously a graduate assistant at the University of Pitts-

burgh, has accepted a position as technical engineer at I.B.M. Corporation, Endicott, New York.

Dr. M. A. Shader of Stanford University is now an applied science representative with I.B.M. Corporation, San Francisco, California.

Sister M. Bibiana, formerly an instructor at the College of St. Teresa, Winona, Minnesota, is now at Cotter High School, Winona, Minnesota.

Professor Emeritus J. B. Coleman of the University of South Carolina died in Vienna, Austria, on February 21, 1955. He was a charter member of the Association.

Professor L. S. Johnston of the University of Detroit died on February 18, 1955. He was a member of the Association for thirty-eight years. The Johnston family is establishing the Leon S. Johnston Prize in Mathematics at the University of Detroit. Friends of Professor Johnston who may wish to make a contribution to the fund may write to the Chairman of the Department of Mathematics of the University.

THE MATHEMATICAL ASSOCIATION OF AMERICA

NEW MEMBERS

Official Reports and Communications

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 118 persons have been elected to membership by the Board of Governors on applications duly certified.

BRIAN ABRAHAMSON, S.M. (Chicago) Lecturer, University of Cape Town, South Africa.

R. H. ACKERSON, A.M. (Columbia U.) Teaching fellow, Alabama Polytechnic Institute.

H. G. APOSTLE, Ph.D. (Harvard) Chairman, Department of Mathematics, Grinnell College.

LT. COL. J. W. AULT, M.A. (Ohio S.U.) Asso. Professor, U. S. Air Force Academy, Denver, Colo.

MRS. JOAN L. AUSTRA, A.B. (Montclair S.T.C.) Teacher, Florence Avenue School, Irvington, N. J.

C. H. BAILEY, M.A. (W. Va. U.) Asso. Professor, Concord College.

R. G. BARTLE, Ph.D. (Chicago) Instr., Yale University.

R. V. BENSON, A. M. (Southern Calif.) Instr., Los Angeles City College.

C. E. BOULWARE, Ed.D. (Columbia U.) Asso. Professor. North Carolina College.

A. H. BROWN, Ph.D. (Chicago) Instr., Rice Institute.

WINNIE E. BROWN, B.S. (Tennessee P.I.) Instr., Tennessee Polytechnic Institute.

P. L. BUTZER, Ph.D. (Toronto) Asst. Professor, McGill University.

H. C. CARTER, Ph.D. (Missouri) Professor, Mary Washington College.

E. C. CASALE, M.A. (Temple) Instr., Temple University.

LOUIS CHILD, M.A. (North Carolina) New Mexico College of Agriculture and Mechanic Arts.

VIRGINIA CLOVER, A.B. (Augustana C.) Fellow, University of Arizona.

WHITFIELD COBB, A.M. (North Carolina) Asso. Professor, Guilford College.

LAURENCE COOK, M.S. in Ed. (Drake) Teacher, Junior High School, Eau Claire, Wisc.

GRACE V. CRENSHAW, M.A. (Virginia) Asso. Professor, Danville Branch, Virginia Poly-

technic Institute; Head of Department of Mathematics, Averett College.

ARNO CRONHEIM, Ph.D. (Illinois) Instr., Brandeis University.

D. A. DEFELICE, M.S. (Pittsburgh) Instr., Duquesne University.

MARCEL DELCOURTE, C.E. (Louvain) General Manager, Les Assurances du Boerenbond Belge, Louvain, Belgium.

J. M. DORAN, M.A. (George Peabody) Instr., Tennessee Polytechnic Institute.

CARL DUBOVY, B.S. (C.C.N.Y.) Grad. Student, City College of the City of New York.

HELEN M. DUNN, B.S. (Washington) Teacher, West Seattle High School, Seattle, Wash.

M. R. ELIZER, M.A. (Georgia) Asst. Professor, Emory Junior College.

M. P. EPSTEIN, Ph.D. (Columbia U.) Instr., University of California, Berkeley.

CORDELL EVANS, M.Litt. (Pittsburgh) Grad. Asst., University of Pittsburgh.

JACQUELINE P. EVANS, Ph.D. (Radcliffe) Instr., Smith College.

R. F. FARRELL, B.S. (Canisius) Dynamics Analyst, Bell Aircraft Corp., Niagara Falls, N. Y.

BEVERLY R. FERNER, A.M. (Wisconsin) Instr., Ohio University.

EMMA W. GARNETT, M.A. (George Peabody) Instr., Tennessee Polytechnic Institute.

L. L. GAVURIN, S.M. (Brown) Brooklyn, New York.

P. R. GERWITZ, B.A. (Buffalo) Grad. Student, University of Buffalo.

HERBERT GINSBERG, B.A. (St. Lawrence) Numerical Analyst, General Electric Co., Evendale, O.

O. H. GOEHRING, M.A. (Columbia U.) Mathematician, Cornell Aeronautical Lab., Buffalo, N. Y.

BROTHER JAMES FRANCIS GRAY, M.S. (Northwestern) Grad. Student, University of Notre Dame.

R. P. GROBE, B.S. (Montana S.C.) Grad. Student, Montana State College.

CORNELIUS GROENEWOUD, M.A. (Michigan S.C.) Dynamics Engineer, Bell Aircraft Corp., Buffalo, N. Y.

C. J. HANKS, Ed.D (Arkansas) Instr., California State Polytechnic College.

H. L. HATTER, Ph.D. (Purdue) Math. Statistician, Wright-Patterson Air Force Base.

K. M. HERSTEIN, B.S. (Columbia U.) President, Herstein Laboratories, New York, N. Y.

J. H. HODGES, Ph.D. (Duke) Instr., Duke University.

A. V. HOUGHTON III, M.S. (Illinois) Asst. Professor, Bradley University.

J. L. HOWELL, Ph.D. (Yale) Instr., University of Delaware.

C. W. HUNTER, M.A. (Boston C.) 1st Lieutenant, U.S.M.C., Camp Lejeune, N. C.

PATRICIA A. INMAN, B.A. (Reed) Grad. Student and Research Asst., University of Oregon.

W. H. JONES, M.S. (Chicago) Mathematician, Department of Defense, Washington, D. C.

J. D. KAPLAN, Student, Rutgers University.

CAROLYN E. KAPPEL, Student, Carleton College.

J. E. KELLEY, M.S. (Marquette) Instr., Marquette University.

H. P. KERFOOT, Ph.D. (Southern Calif.) Lecturer, University of Southern California.

M. Y. KITAMURA, Student, University of Hawaii.

ANN E. KLEIN, M.A. (Wichita) Instr., University of Wichita.

D. L. KLIPPENSTEIN, M.S. (Iowa S.C.) Instr., Bethel College.

LOUISE M. KNIFLEY, M.A. (Kentucky) Instr., University of Tennessee, Martin Branch.

R. R. KORFHAGE, B.S.E. (Michigan) Grad. Student, University of Michigan.

E. C. KOVACS, B.S. (Washington & Jefferson) Asst. Professor, University of Pittsburgh.

G. R. KUHN, B.S. (St. Joseph's C.) Teacher, Academy of the Sacred Heart, St. Louis, Mo.

M. A. LAFRAMBOISE, M.A., M.S. (Michigan) Instr., University of Detroit.

R. W. LANZKRON, M.S. (Wisconsin) Fellow, University of Wisconsin.

D. M. LEVY, B.A. (William & Mary) Teacher, Chuckatuck High School, Virginia.

D. R. LEWIS, M.A. (Minnesota) Asst. Professor, State Teachers College, Mankato, Minn.

Cpl. R. L. LIBOFF, B.S. (Brooklyn) Physicist, Army Chemical Center, Md.

SHEN LIN, M.A. (Ohio S.U.) Asst. Instr., Ohio State University.

B. W. LINDGREN, Ph.D. (Minnesota) Instr., University of Minnesota.

MICHAEL LIONE, M.S. (N.Y.U.) Instr., Newark College of Engineering.

R. J. LOCKHART, B. A. (Western Ontario) Lecturer, University of Manitoba.

CAPT. W. D. MARSLAND, JR., M.A. (N.Y.S. College for Teachers at Albany) Instr., U. S. Air Force Academy, Denver, Colo.

EARL MAXIE, M.S. (Tuskegee) Teacher, Grambling College.

C. W. MCARTHUR, Ph.D. (Tulane) Asst. Professor, Alabama Polytechnic Institute.

L. F. MCAULEY, Ph.D. (North Carolina) Instr., University of Maryland.

GRANVILLE MCCORMICK, M.A. (Oregon) Teaching Asst., University of Washington.

NEILL MCSHANE, Student, University of Virginia.

V. J. MONACELLA, B.S. (Gannon) Mathematician, David Taylor Model Basin, Washington, D. C.

J. G. MOSER, Student, Rose Polytechnic Institute.

H. L. NEWMAN, M.S. in E.E. (Buffalo) Sylvania Electric Products, Buffalo, N. Y.

E. D. NICHOLS, M.A. (Illinois) Instr., University of Illinois.

D. A. PAGE, A.M. (Illinois) Instr., University of Illinois.

R. W. PAUL, JR., Student, California Institute of Technology.

R. S. PIERCE, Ph.D. (Calif. I.T.) Research Asst., Harvard University.

A. G. POORMAN, B.A. (Ashland) Grad. Asst., Iowa State College.

FRANK PROPP, M.A. in Ed. (C.C.N.Y.) Teacher, Morris High School, New York, N. Y.

T. D. RINEY, Ph.D. (Purdue) Member, Technical Staff, Bell Telephone Labs., Allentown, Pa.

T. P. ROBINSON, M.S. (Oklahoma A. & M.) Grad. Student, Oklahoma Agricultural and Mechanical College.

J. N. ROGERS, Student, Butler University.

JACK ROSEMAN, B.A. (Boston U.) Grad. Student, University of Massachusetts.

F. W. SCHNEIDER, Columbus, Ohio.

M. A. SHADER, Ph.D. (Syracuse) Applied Science Representative, I.B.M. Corp., San Francisco, Calif.

E. T. SHEFFIELD, M.A. (Minnesota) Physicist, U. S. Naval Radiological Defense Lab., San Francisco, Calif.

F. C. SHERBURNE, JR., B.S. (Toledo) Grad. Asst., Michigan State College.

G. J. SIMMONS, Research Asst., Sandia Corp., Albuquerque, N. M.

SISTER ANITA SIBILIA, A.B. (Montclair S.T.C.) Teacher, St. Joseph's High School, Hammonton, N. J.

SISTER M. IGNATIA, M.S. (Catholic) Head, Department of Mathematics, Marygrove College.

SISTER M. THEODORA, M.S. (Marquette) Instr., Ursuline College.

SISTER MARY ANDREA, Ph.D. (Catholic) Head, Department of Mathematics, St. Mary College, Xavier, Kansas.

SISTER MARY DE SALES, Student, Georgetown University.

H. L. SLONECKER, JR., M.A. (George Peabody) Grad. Student, Vanderbilt University.

J. L. SMITH, B.A. (Louisville) Grad. Asst., University of Pittsburgh.

MAJ. N. H. SMITH, M.S. (S.U. of Iowa) Instr., U. S. Naval Academy.

J. R. STOCK, Dr.Sc. (Swiss Federal I.T.) Mathematician, National Carbon Research Labs., Cleveland, O.

R. T. STUBBS, B.S. (Georgia I.T.) Savannah, Georgia.

E. J. THOMAS, M.A. (Ohio S.U.) Asst. Professor, Mississippi Southern College.

HOWARD TINSLEY, B.S. (Tennessee P.I.) Instr., Tennessee Polytechnic Institute.

R. N. WALTER, Ed.D. (Columbia U.) Teacher, Manhattan High School of Aviation Trades.

R. M. WARTEN, Student, Brooklyn College.

E. C. WATTERS, Ph.D. (Maryland) Asst. Professor, U. S. Naval Academy.

D. F. WEHN, Student, Brooklyn College.

A. M. WHITE, M.A. (U.C.L.A.) Instr., University of Santa Clara.

MRS. MARIE S. WILCOX, M.A. (Indiana) Head, Department of Mathematics, Thomas Carr Howe High School, Indianapolis, Indiana.

J. C. WILSON, M.S. (S.U. of Iowa) Instr., Central College, Pella, Iowa.

MRS. ANNA L. WILSTED, B.S. (Edinboro S.T.C.) Teacher, Townville Consolidated School, Pa.

SAMUEL WOLF, M.S. (Case I.T.) Engineer, Westinghouse Electric Corp., Baltimore, Md.

MRS. DOROTHY WOLFE, M.A. (Wayne) Asst., University of Pennsylvania.

F. B. WRIGHT, JR., Ph.D. (Chicago) Instr., Tulane University.

R. E. WYLLYS, B.A. (Arizona S.C.) Mathematician, National Security Agency, Washington, D. C.

HIDEHIKO YAMABE, Dr.Sc. (Osaka U.) Asst. Professor, University of Minnesota.

W. A. YERKES, Student, Lebanon Valley College.

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirty-second annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the Buena Vista Hotel, Biloxi, Mississippi, with Mississippi Southern College the host institution, on February 18 and 19, 1955. Professor J. W. McClimans, Chairman of the Section, presided at the evening session and Professor P. K. Rees presided at the Saturday morning session.

There were 92 persons registered, including the following 45 members of the Association.

Ruth E. Biggers, Ermon H. Bryant, N. A. Childress, W. H. Cleveland, M. P. Dossey, W. L. Duren, Jr., Elizabeth Freas, L. M. Garrison, M. E. Gillis, A. L. Gilmore, Jr., G. W. Hiller, H. G. Jacob, Jr., John Jones, Jr., H. T. Karnes, Margaret M. LaSalle, Z. L. Loflin, A. C. Maddox, J. W. McClimans, Betty McKnight, R. A. Miller, Benjamin Ernest Mitchell, T. F. Mulcrone, J. D. Munn, S. B. Murray, Arthur Ollivier, Judith M. Pillow, P. K. Rees, L. A. Rife, W. M. Sanders, W. C. Sangren, H. F. Schroeder, Fariebee B. Self, S. W. Shelton, Jr., P. K. Smith, S. M. Spencer, Jr., R. A. Stokes, V. B. Temple, W. B. Temple, E. J. Thomas, Earl Thomas, W. E. Timon, Jr., B. B. Townsend, P. M. Tullier, Jr., B. O. Van Hook, Dale Woods.

The following officers were elected for the coming year: Chairman, Professor T. L. Reynolds, Millsaps College; Louisiana Vice-Chairman, Mr. P. L. Ford, McNeese State College; Mississippi Vice-Chairman, Professor John Jones, Jr., Mississippi Southern College; Secretary-Treasurer, Professor Z. L. Loflin, Southwestern Louisiana Institute.

The invited speaker for the meeting was Dr. W. C. Sangren, Oak Ridge National Laboratory. His address on Friday evening was entitled "The ORACLE, and Other Computers, as a Scientific Tool" and at the Saturday morning session, "Numerical Calculations Associated with the Membrane Problem."

The following papers were presented:

1. *On the triangular form of certain matrices*, by Professor John Jones, Jr., Mississippi Southern College.

I. Schur has shown that if A is a matrix with elements in the field of complex numbers there exists a unitary matrix U such that U^*AU is triangular with U^* the conjugate transpose of U. The purpose of this note is to obtain the triangular form of a class of matrices which have elements that are functions of a real variable x defined on an interval $a \leqq x \leqq b$.

2. *Stieltjes integral representation of convex functions*, by Professor W. B. Temple, Louisiana Polytechnic Institute.

The main result of this paper is contained in the following theorem: Let $f(x)$ be continuous and convex of order k, $(k \geqq 2)$ in $a<x<b$, $a<0$ and $b>1$. Then there exists a non-decreasing function $\mu(x)$ and a polynomial of degree not more than $k-1$ determined by the conditions $f(1)=g(1)$,

$f(0)=g(0), f'(0)=g'(0), \cdots, f^{k-2}(0)=g^{k-2}(0)$, such that

$$f(x) = g(x) - \int_0^1 G_k(x, t)d\mu(t)$$

where

$$G_k(x, t) = \frac{1}{(k-1)!}\left[(1-t)^{k-1}x^{k-1} - (x-t)^{k-1}\right], \qquad 0 \leqq t \leqq x < 1,$$

$$G_k(x, t) = \frac{1}{(k-1)!}(1-t)^{k-1}x^{k-1}, \qquad 1 \geqq t \geqq x > 0.$$

3. *On the rise of a liquid in a rotating container*, by Professor P. K. Smith, Louisiana Polytechnic Institute.

This short paper deals with the equations concerning the behavior of a liquid under rotation in several types of containers. First, the well known fact is derived that a fluid rotating about an axis will assume the form of a paraboloid of revolution. Then for several types of containers the rise of the liquid in the container is shown to be a well-determined function of the angular velocity of the container.

4. *Roses not clovers*, by Professor V. B. Temple, Louisiana College.

Leaves (petals) of roses overlap, leaves of clovers do not. Hence, the curves commonly called roses should be clovers. These curves are defined by writers only by equations of the form $\rho=a \sin k\phi$ and $\rho=a \cos k\phi$. We have defined them geometrically and their general equations are

$$\rho = \frac{a}{n}(n \mp 2) \sin\left[\left(\frac{n}{n \mp 2\lambda}\right)\frac{\pi}{2} \mp \phi\right], \qquad n > 0, \lambda \neq 0.$$

The upper and lower signs represent hypo- and epi-cyclo-roses corresponding to the hypo- and epi-cycloids. See this MONTHLY, vol. 61, 1954, p. 265, and vol. 59, 1952, p. 67. If λ is chosen so that $n/(n-2\lambda)=K$, an integer, these equations reduce to the simple forms above, according as K is even or odd. In this case we have the clovers. If K is not an integer we have the roses.

5. *An exponential Diophantine equation possessing integral solutions*, by Professor W. M. Sanders, Mississippi Southern College.

The exponential Diophantine equation $zx+y=y^z$ for $0<x<z$ and $0<y<z$ is considered. A solution for $x=2$ is obtained.

6. *The vibrations of a circular membrane*, by Professor Wallace Hebert, Louisiana Polytechnic Institute, introduced by the Secretary.

This paper contains a solution of the basic equation of motion of the vibrating membrane with suitable boundary values and initial conditions given. With certain assumptions regarding symmetry the solution can be obtained.

The basic equation of motion of the membrane is given in the form:

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where ∇^2 is the Laplacian operator. Since we wish to consider a membrane of circular form, such as a drum, the equation can conveniently be written in polar co-ordinates, which then becomes

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + k^2v = 0.$$

The solution of the above equation can be shown to be

$$v = AJ_0(kr) + BY_0(kr)$$

where J_0 and Y_0 are the Bessel functions of the zeroth order and of the first and second kind.

7. *Vector duality*, by Professor Benjamin Ernest Mitchell, University of Mississippi.

The vector equation $\lambda\rho = c$ says that c is the power of the common initial point 0 of λ and ρ with respect to a circle constructed on AB as a diameter where A and B are the termini of λ and ρ respectively—and vice versa, according to Clifford. This circle is orthogonal to the circle center O and radius $\sqrt{c}$. If we pin down λ the terminus of ρ will trace a line, the polar of A with respect to circle $(0, \sqrt{c})$. We therefore call λ a line vector (not a line-bound vector). If we pin down ρ and release λ there results a pencil of lines through the terminus of ρ, hence the duality. We therefore call ρ a point vector. The relationship is extended to reciprocation or polar duality (Graustein). The principle may be extended to higher dimensions with the proper choice of dual elements.

8. *On some groups associated with finite geometries*, by Professors John Jones, Jr., W. M. Sanders, and L. A. Rife, Mississippi Southern College.

L. A. Rife established the existence of a finite projective geometry of 57 points in a thesis in 1947, *On Some Finite Geometries*, University of Nebraska (unpublished thesis). Marshall Hall, Jr., has established the uniqueness of the projective plane with 57 points. Using the postulates of O. Veblen and W. H. Bussey to construct finite projective geometries, permutation groups associated with their finite projective geometries are considered.

9. *An affine definition of pi*, by Professor W. L. Duren, Jr., Tulane University.

The customary definition of π is formulated in the euclidean geometry of the plane. Despite the non-invariance of arc length in the plane represented as the Cartesian product of affine line and affine line, the number π is defined naturally in this geometry by means of an ellipse. In fact it is defined also by means of an invariant measure.

10. *A note on the Poncelet line*, by Reverend T. F. Mulcrone, St. Charles College.

This paper is concerned with the form assumed by various theorems involving the Simson or pedal line of a point relative to a triangle when extension is made to the Poncelet line.

11. *The ORACLE, and other computers, as a scientific tool*, by Dr. W. C. Sangren, Oak Ridge National Laboratory.

The last ten years have seen the viewpoint towards computations change radically because of the advent of high speed machines. This should be expected in any field where an improvement of high order, say 10^4 or more, has suddenly taken place. The improvement is in the length of time needed to perform certain mathematical operations. It should be emphasized that the need for mathematics and mathematicians is greatly increased because of the increasing use of machines. The ORACLE and the Mathematical Panel at Oak Ridge are examples of a high speed digital machine and a group which use such machines.

12. *Numerical calculations associated with the membrane problem*, by Dr. W. C. Sangren, Oak Ridge National Laboratory.

One of the problems that arises in nuclear reactor theory is the finding of the fundamental eigenvalue and eigenfunction associated with the classical membrane problem with a fixed boundary. The membrane problem may be characterized either through a differential equation system or by a variational problem. There exist basically two mathematical techniques, analytical and

numerical, for obtaining the desired value and function associated with the given geometry. Since the analytical techniques have been fairly well exploited, the numerical techniques show the greatest future promise. A number of programs or codes have been prepared and run on high speed digital machines, primarily the ORACLE, with the objective of obtaining the desired value and function for odd-shaped membranes. Among the peculiar shaped regions being investigated are L-shapes, T-shapes, crosses and square doughnuts.

Z. L. LOFLIN, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirty-fifth regular meeting of the Southern California Section of the Mathematical Association of America was held at Santa Monica City College on March 12, 1955. Professor E. C. Rex, Chairman of the Section, presided.

The attendance was 105, including the following 83 members of the Association:

L. J. Adams, Frances Campbell Amemiya, A. R. Amir-Moez, Norman Anning, T. M. Apostol, Leon Bankoff, J. L. Barnes, Mabel S. Barnes, Lulu Bechtolsheim, May M. Beenken, Clifford Bell, J. S. Bendat, R. V. Benson, L. T. Black, H. F. Bohnenblust, R. E. Bruce, Jessie R. Campbell, L. M. Coffin, E. P. Coleman, P. H. Daus, R. A. Dean, C. R. DePrima, R. P. Dilworth, G. M. Eddington, L. R. Ford, Jr., G. E. Forsythe, Ruth M. Frisch, W. V. Gamzon, W. H. Glenn, Jr., B. K. Gold, Jr., Edison Greer, Nathaniel Grossman, C. J. A. Halberg, Jr., W. L. Hart, V. C. Harris, A. R. Harvey, H. L. Hendricks, R. B. Herrera, R. E. Horton, D. H. Hyers, C. G. Jaeger, P. B. Johnson, G. R. Kaelin, Rosella Kanarik, P. J. Kelly, G. J. Kleinhesselink, L. C. Lay, Fred Marer, G. F. McEwen, E. E. Moots, F. R. Morris, A. B. Neale, J. B. Nelson, L. J. Paige, R. W. Paul, Jr., D. J. Peterson, R. R. Phelps, H. R. Pyle, F. C. Reed, E. C. Rex, J. M. Robb, E. M. Scheuer, Sister M. Madeleine Rose, Abe Sklar, Samuel Skolnik, N. B. Smith, T. H. Southard, Maria Weber Steinberg, A. C. Sugar, T. E. Sydnor, P. Y. Tani, Elmer Tolsted, C. B. Tompkins, C. W. Trigg, S. E. Urner, F. A. Valentine, R. J. Walker, Morgan Ward, P. A. White, R. L. White, A. L. Whiteman, B. R. Wicker, A. D. Wirshup.

At the business meeting the following officers were elected for the next academic year: Chairman, Professor S. E. Urner, Los Angeles State College; Vice-Chairman, Professor H. F. Bohnenblust, California Institute of Technology; Program Committee: Chairman, Professor Elmer Tolsted, Pomona College, Mr. R. B. Herrera, Los Angeles City College, Professor D. V. Steed, University of Southern California, Mr. T. E. Sydnor, Pasadena City College.

The following program was presented:

1. *A role for complex numbers in analytic geometry*, by Professor V. C. Harris, San Diego State College.

By using one plane as both a real plane and a complex plane, the student can graph $y=f(x)$ as a complex function y of a real variable x. If $f(x)$ represents functions of the type usually considered in analytic geometry, little if any additional technique needs to be developed. Alternative methods for this purpose are too difficult or less informative. The result is intended to be a better understanding of the place of complex numbers in mathematics.

2. *Group-varieties*, by Professor Iacopo Barsotti, University of Pittsburgh and University of Southern California, introduced by Professor A. L. Whiteman.

A group-variety is an irreducible algebraic subvariety G of a projective space over a field k, such that there exists a proper subvariety F of G, and a rational law of composition $(P, Q)\rightarrow PQ$ which turns $G-F$ into a group. If F is empty, G is called an abelian variety (necessarily commuta-

tive). If G is commutative but not abelian, it contains a maximal rational group-subvariety V, and G/V is abelian; V is a projective space, and the law of composition on V, if k has characteristic zero, is obtained by addition of certain cartesian co-ordinates (as in vector spaces), and multiplication of others. If G is not commutative, and C is its center, G/C is a Vessiot variety (representative of an algebraic group of matrices); moreover, G contains a maximal Vessiot variety V; V is invariant, and G/V is abelian.

3. *The experimental mathematics program at Haverford College*, by Professor P. B. Johnson, Occidental College.

The mathematics program can not be divorced from the entire curriculum at Haverford, where emphasis is on thinking and doing significant things rather than transmitting the culture. The freshman program gives major attention to basic ideas which have proved to be the seeds of modern mathematical thinking. The usual topics are then covered rapidly. A traditional calculus course follows, but this will be revised. A distinguished mathematician is brought in to teach a special topic in modern mathematics from an undergraduate point of view. The rest of the program is traditional. Many elements of the program can be successful elsewhere.

4. *Soap films and minimal surfaces*, by Professor C. R. DePrima, California Institute of Technology.

For more than a decade, R. Courant has employed soap film experiments as a heuristic guide to the study of the general Plateau-Douglas problems and related minimal surface problems. In this lecture the general phenomena of non-uniqueness, discontinuous dependence and instability associated with these minimal surface problems were demonstrated with soap film experiments.

5. *Report on the Los Angeles Mathematics Newsletter*, by Mr. Samuel Skolnik, Los Angeles City College.

The Los Angeles Mathematics Newsletter was initiated at Los Angeles City College in the Fall of 1953. This Newsletter is devoted to the interests of high school students of mathematics. Each senior high school in Los Angeles gets a sufficient number of copies to give to each student in the advanced mathematics classes (beyond plane geometry). During 1953–54 two issues were published. Three issues will be published during the current academic year. The readers of this abstract are invited to send in articles and problems for publication in the Newsletter. The articles should be 100 to 500 words in length on any topic suitable to a good high school mathematics major. Of course, the subject matter need not be new or original.

6. *The minimal cut theorem*, by Dr. L. R. Ford, Jr., RAND Corporation.

In the theory of flows through networks, a cut is a set of arcs whose removal disconnects the source from the sink. The value of a cut is the sum of the capacities of its arcs. The minimal cut theorem, stating that the maximal flow value is equal to the minimal cut value, is proved. Menger's theorem (König, *Theorie der Graphen*, Chelsea, N. Y., 1950, p. 244) is derived as an immediate corollary. A computational procedure for certain types of planar graphs is discussed and illustrated. This paper is a report of the work of the speaker and D. R. Fulkerson.

7. *Change ringing*, by Professor R. J. Walker, Cornell University.

The traditional English method of ringing permutations on a set of church bells leads to several problems in group theory. For example: Let $a_1, \cdots, a_n$ be a set of elements of a finite group G of order n. Is it possible to obtain each element b_i of G by the recursion process $b_{i+1}=a_{j_i}b_i$, $i=1, \cdots, n-1$, $b_1=I$? How small can n be chosen, and how does one determine the sequence j_i? These questions also arise in the problem of generating finite groups in automatic calculating machines.

P. H. DAUS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, McNeese State College, Lake Charles, Louisiana, February 17–18, 1956.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN, University of Michigan, Ann Arbor, March, 1956.
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA
OHIO
OKLAHOMA, Oklahoma City University, October 28, 1955.
PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 17, 1955.
PHILADELPHIA
ROCKY MOUNTAIN
SOUTHEASTERN, University of Georgia, Athens, March 16–17, 1956.
SOUTHERN CALIFORNIA, Pomona College, Claremont, March 10, 1956.
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